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CFT description of identity string field: toward derivation of the VSFT action

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Abstract: We concretely define the identity string field as a surface state and deal with it consistently in terms of conformal field theory language, never using its formal properties nor oscillator representation of it. The generalized gluing and resmoothing theorem provides us with a powerful computational tool which fits into our framework. Among others, we can prove that in some situations the identity state defined this way actually behaves itself like an identity element under the $*$-product. We use these CFT techniques to give an explicit expression of the classical solution in the ordinary cubic string field theory having the property that the conjectured vacuum string field theory action arises when the cubic action is expanded around it.

Keywords: Bosonic Strings, Tachyon Condensation, String Field Theory.

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Contents

1. Introduction

In the last one year, vacuum string field theory (VSFT), which was first proposed by Rastelli, Sen and Zwiebach in [1] as a candidate for the theory which describes the open string dynamics around the tachyon vacuum, has intensively been studied. In the early days, it was recognized [2] that the matter part of the equation of motion reduced to an equation satisfied by any projection operator if we assume that the classical solution representing a D-brane factorized into the ghost part and the matter part, because the kinetic operator $\mathcal{Q}$ of VSFT was supposed to be made purely out of ghost fields. Such projectors, which are squared to itself under the $*$-product of string field theory, have been constructed algebraically as "sliver states" [2, 3] in the matter sector. Without any knowledge of the ghost sector, the ratios of tensions of classical sliver solutions with different dimensions were calculated and were shown to agree numerically with the known results about the ratios of D-brane tensions. After that, the sliver states were reconstructed in terms of boundary conformal field theory [4, 5, 6] and were interpreted as rank-one projection "operators" in the sense that they operated in the space of half-string functionals [7].

Research in the ghost sector has started with the work of Hata and Kawano [8], who showed that it was possible to determine the form of $\mathcal{Q}$ uniquely by requiring the existence of the Siegel gauge solution. This work, written in terms of operator language, was refined by one of us [9] and Okuyama [10]. Another choice of $\mathcal{Q}$ was proposed by Gaiotto, Rastelli, Sen and Zwiebach in [11], where they considered a local ghost insertion at the midpoint of the open string. Surprisingly, it was also shown [11] that these two candidates for $\mathcal{Q}$, which appeared to have different origins, coincided numerically with each other. Although the construction of vacuum string field theory has been completed this way, we are not allowed to compute the absolute values of tensions of the brane solutions[1] because this theory arises as such a singular limit that it needs some regularization for this purpose. Moreover, it seems difficult to reproduce the known physical spectrum of D-branes from the brane solutions in vacuum string field theory. Even though we could specify the "on-shell tachyon state" with mass $m^2 = -1/\alpha'$ [8, 13], it would be a hard task to impose the transversality conditions on the vector (tensor) fields, e.g. $k^\mu A_\mu(k) = 0$, if we start from the kinetic operator $\mathcal{Q}$ which is completely independent of the matter sector and the factorized solutions.

Confronted with these difficulties, we have been led to consider the solutions which contain the BRST operator Q_B or something in them. In fact, such attempts were made in the purely cubic formulation of string field theory[2] [14] about 15 years ago. There, Horowitz etal. showed that it was possible to construct solutions of the form[3] $Q_L(f)\mathcal{I} + c_L(g)\mathcal{I}$ in the original cubic open string field theory (OSFT) such that there were no physical open string excitations around them. However, their arguments crucially depended on the formal properties of the identity string field $\mathcal{I}$, which made the arguments less persuasive.

In this paper, we will consistently deal with the identity string field in terms of the conformal field theory language, never relying on any formal property such as $\mathcal{I} * A = A * \mathcal{I} = A$ for every A. The generalized gluing and resmoothing theorem allows us to carry out concrete calculations fully within our framework. We will show that the identity string field $\mathcal{I}$ defined this way actually acts as the identity element of the $*$-algebra in some situations. Thus, one of the aims of this paper is to establish how to treat the identity state using conformal field theory in a definite manner.

Furthermore, this computational scheme will be used to propose a possible form of the tachyon vacuum solution. More precisely, we consider a specific configuration of the string field including the identity state and the BRST operator, and show that this configuration indeed solves the equation of motion derived from the ordinary cubic string field theory

[1] Hata and Moriyama [12] followed the strategy proposed in [8] that one could determine the tension of the brane solution by calculating the open string coupling constant g_o defined as the on-shell scattering amplitude of three "tachyons" (so-called HK state) living on the brane solution, and suggested that the classical solution found in [8] corresponded to the configuration of two D25-branes from the energetic aspects. Later, it was pointed out in [13] that this claim was not valid on the ground that the authors of [8, 12] had used the equation of motion for their "tachyon state" which in fact had not been satisfied in a strong sense.

[2] Recently Matsuo [15] reconsidered this theory and generalized it to a matrix version to let it be able to describe an arbitrary number of D-branes.

[3] Takahashi and Tanimoto have also considered such solutions in their recent works [16], though their solutions are not related to the problem of tachyon condensation.

action defined on a D25-brane. Expanded around this solution, the action takes the same form as the conjectured vacuum string field theory action! Hence we guess that our solution corresponds to the tachyon vacuum in some singular frame. We further discuss that the deformation of this solution may lead to a regularized version of the VSFT action postulated in [11], though not so conclusive.

This paper is organized as follows. In section 2, we explain our definitions of string field theory vertices and of identity string field in the CFT language, which will be used throughout the rest of this paper. In section 3, we describe our main computational tool, namely the generalized gluing and resmoothing theorem, and derive a general formula for gluing functions. Section 4 is devoted to technical calculations used later. The readers who are interested only in the relation between VSFT and OSFT can skip sections 3 and 4, but should not forget that our calculations have been done based consistently on the CFT prescription. In section 5 we propose a solution which connects OSFT to VSFT. In section 6 we discuss some open problems and present possible future directions.

2. Our framework

2.1 Conformal field theory approach to string field theory

In this paper we will adopt the conformal field theory description of the string field theory vertices [17]. In this formulation, the BPZ inner product and the 3-string vertex are defined as

$$\langle A, B\rangle = \langle I\circ A(0)\ B(0)\rangle_{\rm UHP}\,, \tag{2.1}$$

$$\langle A, B*C\rangle = \left\langle f_1^{(3)}\circ A(0)\ f_2^{(3)}\circ B(0)\ f_3^{(3)}\circ C(0)\right\rangle_{\rm UHP}\,, \tag{2.2}$$

respectively, where $f\circ\mathcal{O}(0)$ denotes the conformal transform of the vertex operator $\mathcal{O}(0)$ by the conformal map f, $\langle\ldots\rangle_{\rm UHP}$ is the correlation function among the vertex operators evaluated on an upper half-plane (UHP), and the conformal maps are defined by

$$I(z) = -\frac{1}{z}\,, \qquad \text{(inversion)}\,, \tag{2.3}$$

$$f_1^{(3)}(z) = h^{-1}\left(e^{-2\pi i/3}h(z)^{2/3}\right), \tag{2.4}$$

$$f_2^{(3)}(z) = h^{-1}\left(h(z)^{2/3}\right), \tag{2.5}$$

$$f_3^{(3)}(z) = h^{-1}\left(e^{2\pi i/3}h(z)^{2/3}\right), \tag{2.6}$$

where we have for convenience defined an $\mathrm{SL}(2,\mathbb{C})$ map $h(z)$ through

$$h(z) = \frac{1+iz}{1-iz}\,, \qquad h^{-1}(z) = -i\frac{z-1}{z+1}\,. \tag{2.7}$$

h takes an upper half-plane ($\mathrm{Im}\, z\geq 0$) to the inside of a unit disk ($|h(z)|\leq 1$) in a one-to-one way. The state-operator isomorphism has been used to map any state $|\mathcal{O}\rangle$ to the corresponding vertex operator $\mathcal{O}(0)$ (with a slight abuse of notation) via the relation $|\mathcal{O}\rangle = \mathcal{O}(0)|0\rangle$, where $|0\rangle$ denotes the $\mathrm{SL}(2,\mathbb{R})$-invariant vacuum. More detailed explanations are found in [17, 25].

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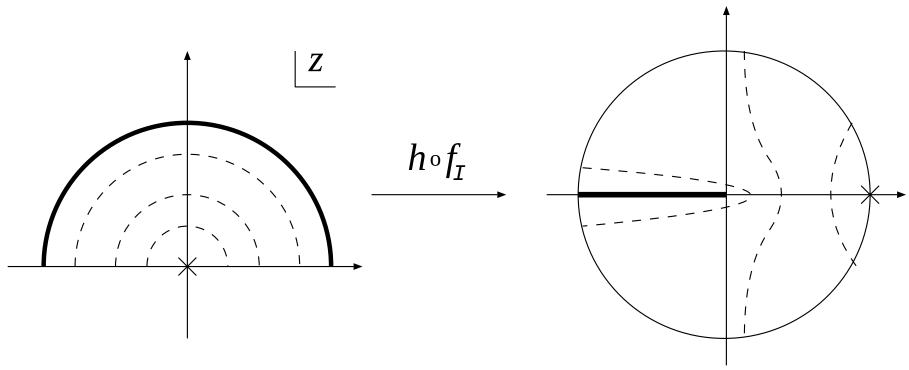

Figure 1: Realization of the left-right overlap by means of a conformal map $h \circ f_{\mathcal{I}}$.

2.2 Defining the "identity" string field

Instead of taking the state $|\mathcal{I}\rangle$ to be an identity element of the $*$-algebra of string field theory, we will throughout define it by the relation

$$\langle \mathcal{I}, \mathcal{O} \rangle = \langle f_{\mathcal{I}} \circ \mathcal{O}(0) \rangle_{\mathrm{UHP}}\,, \tag{2.8}$$

where

$$f_{\mathcal{I}}(z) = h^{-1}\left(h(z)^2\right), \tag{2.9}$$

in terms of the conformal field theory language. The reason why the state $\langle \mathcal{I}|$ defined this way may be considered as the "identity" element of open string field theory is that the conformal map $f_{\mathcal{I}}(z)$ geometrically realizes the overlap of left- and right-halves of the open string (figure 1), originally referred to as the integration operation $\int \Phi$ for an open string field in [18]. Then it perfectly makes sense to ask whether or not $\mathcal{I}$ really behaves itself like the identity element under the $*$-multiplication. In fact, we will show in section 4 that $\mathcal{I} * A = A * \mathcal{I} = A$ certainly holds true for a large class of states A.

In general, a wedge state $\langle n|$ of an angle $2\pi/n$ was defined in [19] as

$$\langle n, \mathcal{O} \rangle = \left\langle f^{(n)} \circ \mathcal{O}(0) \right\rangle_{\mathrm{UHP}}, \tag{2.10}$$

with

$$f^{(n)}(z) = h^{-1}\left(h(z)^{2/n}\right). \tag{2.11}$$

Hence, our definition of $\langle \mathcal{I}|$ coincides with that of the wedge state $\langle n = 1|$ of an angle 2π.

For convenience, here we collect some results which will be obtained in section 4. Below, $\langle \phi|$ and $|\psi\rangle$ denote arbitrary Fock space states, i.e. those which can be created by the action of local vertex operators on the $\mathrm{SL}(2,\mathbb{R})$-invariant vacuum as $\langle \phi| = \langle 0|I \circ \phi(0)$, $|\psi\rangle = \psi(0)|0\rangle$. $*$-multiplication formulae including $|\mathcal{I}\rangle$ are:

$$\begin{aligned} \langle \phi, \mathcal{I} * \psi \rangle &= \langle \phi, \psi * \mathcal{I} \rangle = \langle \phi, \psi \rangle\,, \\ \langle \phi, \mathcal{I} * \mathcal{O}\mathcal{I} \rangle &= \langle \phi, \mathcal{O}\mathcal{I} * \mathcal{I} \rangle = \langle \phi, \mathcal{O}\mathcal{I} \rangle\,, \end{aligned} \tag{2.12}$$

where $\mathcal{O}$ represents a local vertex operator or a contour-integrate of it. From these equations, $\mathcal{I}$ actually looks like an identity element of the $*$-algebra when we restrict ourselves

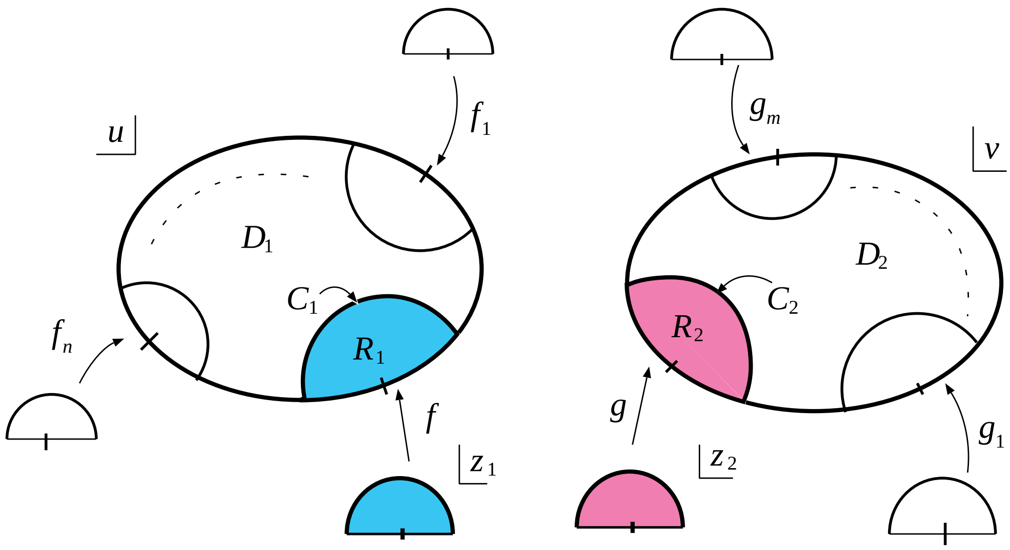

Figure 2: Conformal maps f's and g's from upper half-disks to global disks $\mathcal{D}_1, \mathcal{D}_2$.

to the above settings. The BRST operator Q_B, which governs the perturbative behavior of open strings living on a D25-brane, satisfies the following relations:

$$\{Q_B, Q_L\} = 0\,, \tag{2.13}$$

$$\langle \mathcal{I}|Q_B|\psi\rangle = \langle \mathcal{I}|(Q_R + Q_L)|\psi\rangle = 0\,, \tag{2.14}$$

$$\langle \phi, (Q_R A) * B + (-1)^{|A|} A * (Q_L B)\rangle = 0\,, \tag{2.15}$$

which have been at the heart of the manipulations in purely cubic string field theory [14]. Q_L, Q_R will be defined in section 4. Equation (2.15) holds for A, B in a class larger than that consisting of the Fock space states.

3. Generalized gluing and resmoothing theorem

In the CFT formulation of string field theory, it is helpful to make use of so-called generalized gluing and resmoothing theorem (GGRT), which has been developed in [17, 20, 21], as a computational scheme. After reviewing the geometrical aspects of the gluing theorem following [20] in subsection 3.1, we will derive a general formula for conformal mappings from the original surfaces to the sewed surface, paying great attention to the problem of precisely defining the range of angles.

3.1 Geometrical description of the sewing procedure

Let us suppose that we are given $(n+1)$- and $(m+1)$-point correlation functions on disks[4] $\mathcal{D}_1, \mathcal{D}_2$, respectively, defined by

$$I^{(n+1)}_{\mathcal{D}_1} = \langle f_1 \circ \Phi_{r_1}(0) \cdots f_n \circ \Phi_{r_n}(0)\ f \circ \Phi_r(0)\rangle_{\mathcal{D}_1}\,, \tag{3.1}$$

[4]Similar arguments can be given for the case of spheres.

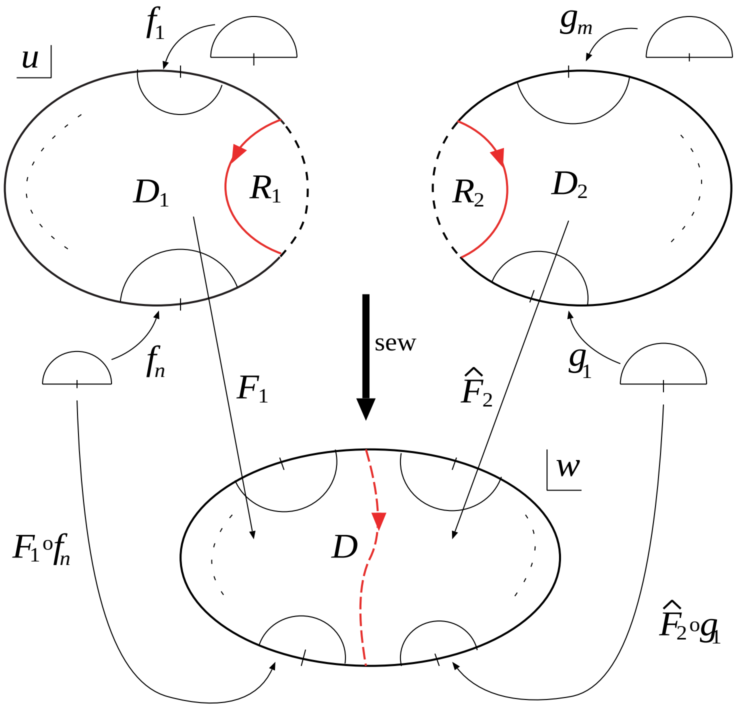

Figure 3: A new disk $\mathcal{D}$ is obtained by sewing the regions $\mathcal{D}_1 - R_1$ and $\mathcal{D}_2 - R_2$ together.

$$I_{\mathcal{D}_2}^{(m+1)} = \langle g_1 \circ \Phi_{s_1}(0) \cdots g_m \circ \Phi_{s_m}(0) \ g \circ \Phi_s(0) \rangle_{\mathcal{D}_2} \,, \tag{3.2}$$

where f's and g's represent conformal maps each of which embeds a unit upper half-disk defining a local coordinate system into the corresponding region in $\mathcal{D}_1$ or $\mathcal{D}_2$ (as shown in figure 2), and Φ's are local vertex operators defined on their own local coordinate disks. Now consider sewing two disks $\mathcal{D}_1$ and $\mathcal{D}_2$ into another disk $\mathcal{D}$ in the following way. First remove the regions R_1 and R_2 which are the images of the unit upper half-disks under the maps f and g, respectively, as indicated in figure 2. Let z_1, z_2 be the coordinates on the unit half-disks corresponding to the regions $f^{-1}(R_1)$, $g^{-1}(R_2)$. Then, sew two surfaces $\mathcal{D}_1 - R_1$ and $\mathcal{D}_2 - R_2$ along the curves C_1 and C_2 ($C_1 = \{f(z_1); |z_1| = 1, \mathrm{Im}\, z_1 \geq 0\}, C_2 = \{g(z_2); |z_2| = 1, \mathrm{Im}\, z_2 \geq 0\}$) through the relation

$$z_1 = I(z_2) = -\frac{1}{z_2} \,. \tag{3.3}$$

Let us call the sewed surface (disk) $\mathcal{D}$, and the global coordinate on it w. The above sewing procedure is shown in figure 3. There we have defined F_1 and $\widehat{F}_2$ to be the maps such that $w = F_1(u)$ and $w = \widehat{F}_2(v)$, where u and v are global coordinates on $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. Looking at the resulting surface $\mathcal{D}$, one may notice that this surface defines an $(n+m)$-point correlation function

$$\begin{aligned} I_{\mathcal{D}}^{(n+m)} = \Big\langle F_1 \circ f_1 \circ \Phi_{r_1}(0) \cdots F_1 \circ f_n \circ \Phi_{r_n}(0) \widehat{F}_2 \circ g_1 \circ \Phi_{s_1}(0) \cdots \\ \cdots \widehat{F}_2 \circ g_m \circ \Phi_{s_m}(0) \Big\rangle_{\mathcal{D}} \,. \end{aligned} \tag{3.4}$$

In fact, the generalized gluing and resmoothing theorem states that the correlation function $I_{\mathcal{D}}^{(n+m)}$ on the sewed surface $\mathcal{D}$ can indeed be obtained by contracting the correlation functions $I_{\mathcal{D}_1}^{(n+1)}, I_{\mathcal{D}_2}^{(m+1)}$ on the original surfaces $\mathcal{D}_1, \mathcal{D}_2$ using the "metric"

$$h^{rs} = \langle \Phi_r^c, \Phi_s^c \rangle = \langle I \circ \Phi_r^c(0) \ \Phi_s^c(0) \rangle \tag{3.5}$$

as

$$I_{\mathcal{D}}^{(n+m)} = \sum_{r,s} I_{\mathcal{D}_1}^{(n+1)}(\Phi_{r_1}, \ldots, \Phi_{r_n}, \Phi_r) I_{\mathcal{D}_2}^{(m+1)}(\Phi_{s_1}, \ldots, \Phi_{s_m}, \Phi_s) h^{rs} \,. \tag{3.6}$$

In eq. (3.5), $\{\langle \Phi_r^c |\}$ denotes the set of bra-states which is dual to the complete set $\{|\Phi_r\rangle\}$ of ket-states in the sense that

$$\langle \Phi_r^c, \Phi_s \rangle = \delta_{rs} \,. \tag{3.7}$$

More explicitly, we have

$$\begin{aligned}
&\sum_r \langle f_1 \circ \Phi_{r_1}(0) \cdots f_n \circ \Phi_{r_n}(0) \ f \circ \Phi_r(0) \rangle_{\mathcal{D}_1} \langle g_1 \circ \Phi_{s_1}(0) \cdots g_m \circ \Phi_{s_m}(0) \ g \circ \Phi_r^c(0) \rangle_{\mathcal{D}_2} = \\
&\quad = \left\langle F_1 \circ f_1 \circ \Phi_{r_1}(0) \cdots F_1 \circ f_n \circ \Phi_{r_n}(0) \ \widehat{F}_2 \circ g_1 \circ \Phi_{s_1}(0) \cdots \widehat{F}_2 \circ g_m \circ \Phi_{s_m}(0) \right\rangle_{\mathcal{D}} ,
\end{aligned} \tag{3.8}$$

for a theory with vanishing total central charge. Before completing the gluing procedure, we have to construct the conformal mappings F_1 and $\widehat{F}_2$ explicitly. Since $\mathcal{D}_1 - R_1$ and $\mathcal{D}_2 - R_2$ are glued together along the curves C_1 and C_2 with the identification (3.3), the image of a point $P(z_1)$ on the arc $\{|z_1| = 1, \mathrm{Im}\, z_1 \geq 0\}$ under the map $F_1 \circ f$ must coincide with the image of the corresponding point $z_2 = -1/z_1$ under the map $\widehat{F}_2 \circ g$ in the w-plane (figure 4). This condition is expressed as

$$F_1 \circ f(z) = \widehat{F}_2 \circ g \circ I(z) \tag{3.9}$$

around the joint curve $\{|z| = 1, \ \mathrm{Im}\, z \geq 0\}$. In the next subsection, we shall explain how to find such functions $F_1, \widehat{F}_2$ for given f, g of the special form.

3.2 Constructing F_1 and $\widehat{F}_2$ for wedges

For the case of string field theory, we often encounter the situation in which the conformal maps f and g take the form

$$f(z) = h^{-1}\left(e^{i\eta_1} h(z)^{\gamma_1}\right), \qquad g(z) = h^{-1}\left(e^{i\eta_2} h(z)^{\gamma_2}\right), \tag{3.10}$$

where $h(z)$ is defined in (2.7). The function $h \circ f$ maps a unit upper half-disk to a wedge R_1 of an angle $\pi\gamma_1$ bounded by a unit circle. The global interaction disk $\mathcal{D}_1$ (to be more precise we should write it as $h(\mathcal{D}_1)$, but we will omit h below) in this case is represented by a unit disk. Hence the region $\mathcal{D}_1 - R_1$ complementary to the wedge R_1 in $\mathcal{D}_1$ also takes the form of a wedge of an angle $2\pi - \pi\gamma_1$. The same as above holds for the regions $\mathcal{D}_2, R_2$ if we replace f, γ_1 by g, γ_2, respectively. According to the strategy described in the last subsection, we will sew these two wedges $\mathcal{D}_1 - R_1$ and $\mathcal{D}_2 - R_2$ together to make up a new unit disk $\mathcal{D}$ with no conical singularities. The rôle of this sewing procedure is played by the

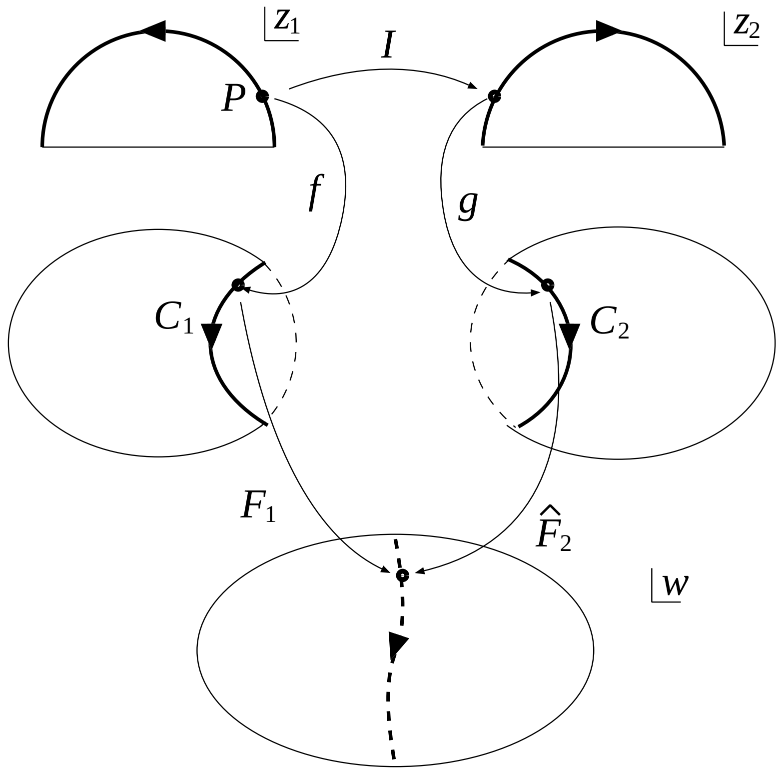

Figure 4: The images of a point P under the maps $F_1 \circ f$ and $\widehat{F}_2 \circ g \circ I$ must agree in the w-plane.

conformal maps F_1 and $\widehat{F}_2$. Since both of $\mathcal{D}_1 - R_1$ and $\mathcal{D}_2 - R_2$ are wedges, we expect that desired maps $F_1, \widehat{F}_2$ can be obtained by combining the rigid rotations of the wedges around the origin with the changes of angles of the wedges. Hence we put the following ansatz

$$F_1(z) = h^{-1}\left(e^{i\phi}h(z)^{\alpha}\right), \qquad \widehat{F}_2(z) = h^{-1}\left(e^{i\theta}h(z)^{\beta}\right), \tag{3.11}$$

and determine the values of $\alpha, \beta, \phi, \theta$ in such a way that the eq. (3.9) should be satisfied.

First we consider the map $F_1 \circ f(z_1) = h^{-1}(e^{i\phi}(e^{i\eta_1}h(z_1)^{\gamma_1})^{\alpha})$, referring to figure 5. In the following, we will be watching how the left half QA of the string (indicated by a bold arrow) moves. Under the map $\widehat{u} = e^{i\eta_1}h(z_1)^{\gamma_1}$, the local coordinate disk represented by a unit upper half-disk in the z_1-plane is mapped to the region R_1 in the $\widehat{u}$-plane, and left half-string QA is mapped to a line segment the angle of which is $-\frac{\pi}{2}\gamma_1 + \eta_1$ to the positive real axis. When we come to consider taking the complement of R_1, ambiguity begins to arise in the definition of angles. Since the difference 2π in angle on the $\widehat{u}$-plane is expanded (or compressed) to $2\pi\alpha$ on the $\widehat{w}$-plane (defined below), this ambiguity could affect the results seriously. We will follow the convention that the angle of the line segment QA is common to the two regions R_1 and $\mathcal{D}_1 - R_1$, whereas the angle of the line segment QB seen from the complement $\mathcal{D}_1 - R_1$ is diminished by 2π as compared to that of QB seen from R_1 (as shown in figure 5). Then the range $\mathcal{D}_1 - R_1$ of definition of the map $F_1 \circ h^{-1}$ is unambiguously fixed to the sector bounded by the two lines of angles $(-\frac{\pi}{2}\gamma_1 + \eta_1)$ and $(\frac{\pi}{2}\gamma_1 + \eta_1 - 2\pi)$ and by a unit circle $|\widehat{u}| = 1$. The angle $\angle AQB$ of this sector is given

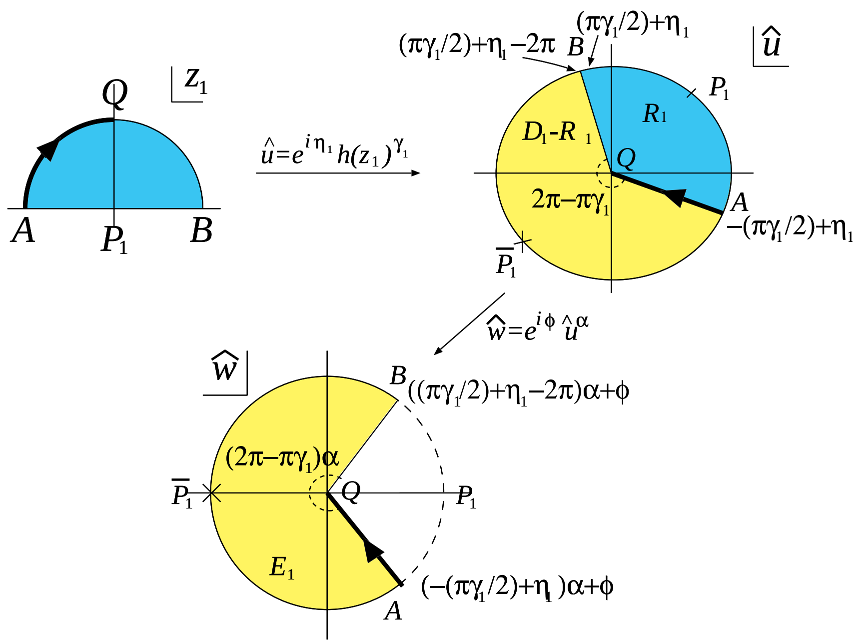

Figure 5: Geometrical construction of the map $F_1 \circ f(z)$.

by $2\pi - \pi\gamma_1$. Let us mark the point in the middle of the arc AB in $\mathcal{D}_1 - R_1$ as $\overline{P_1}$, which is diametrically opposite to the insertion point P_1 on $\mathcal{D}_1$ of the local vertex operator Φ_r associated with the map f in eq. (3.1). As the next step, we perform the mapping $\widehat{w} = e^{i\phi}\widehat{u}^{\alpha}$. Making use of the degree of freedom of ϕ, we take $\overline{P_1}$ to the point $e^{-i\pi}$ on the $\widehat{w}$-plane. This procedure of fixing the phase of the point $\overline{P_1}$ is not essential for the gluing process because only the *relative* position between two regions $\mathcal{D}_1 - R_1$ and $\mathcal{D}_2 - R_2$ on the $\widehat{w}$-plane becomes important. Nonetheless, we have done it because it slightly facilitates the treatment of angles. In fact, this additional degree of freedom of rotating the unit disk can always be provided by the $\mathrm{SL}(2,\mathbb{R})$ transformation inside the correlator (3.4) so that it obviously makes no difference to the final results. Let us call the image of the region $\mathcal{D}_1 - R_1$ under the map $\widehat{u} \mapsto \widehat{w}$ the region E_1. The geometrical data we have obtained are:

$$\text{The angle of the sector } E_1 : (2\pi - \pi\gamma_1)\alpha\,, \tag{3.12}$$

$$\text{The angle of the line segment } QA : \left(-\frac{\pi}{2}\gamma_1 + \eta_1\right)\alpha + \phi \in [-\pi, 0]\,, \tag{3.13}$$

$$\text{The angle of the line segment } QB : \left(\frac{\pi}{2}\gamma_1 + \eta_1 - 2\pi\right)\alpha + \phi \in [-2\pi, -\pi]\,. \tag{3.14}$$

The restrictions on the angles follow from the choice $\widehat{w}(\overline{P_1}) = e^{-\pi i}$. Moreover, it would be obvious that the angle $\angle AQP_1$ is equal to the angle $\angle BQP_1$. This condition is algebraically written as

$$\left(\frac{\pi}{2}\gamma_1 + \eta_1 - 2\pi\right)\alpha + \phi + 2\pi = -\left\{\left(-\frac{\pi}{2} + \eta_1\right)\alpha + \phi\right\}, \tag{3.15}$$

which could be understood if one remembers the definition of angles on the $\widehat{w}$-plane.

In the same way as above, we construct the other map $\widehat{F}_2 \circ g \circ I(z) = h^{-1}(e^{i\theta}(e^{i\eta_2}h(I(z))^{\gamma_2})^{\beta})$ following figure 6. Since the gluing will be performed with the iden-

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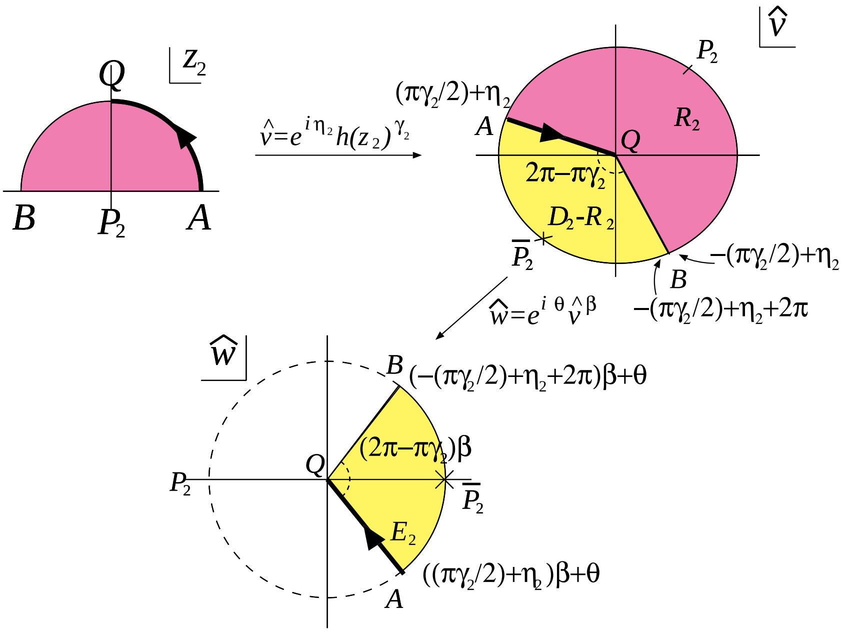

Figure 6: Geometrical construction of the map $\widehat{F}_2 \circ g \circ I(z)$.

tification $z_2 = I(z_1) = -1/z_1$, left and right of the string should be reversed on the arc $\{|z_2| = 1, \mathrm{Im}\, z_2 \geq 0\}$ as compared to the previous case (z_1-plane). Under the map $\widehat{v} = e^{i\eta_2} h(z_2)^{\gamma_2}$, the unit upper half-disk is mapped to the region R_2. In the same sense as in the case of the disk $\mathcal{D}_1$, we keep fixed the angle of the line segment QA to the positive real axis during the process of taking the complement of R_2 in $\mathcal{D}_2$. Then the angle of QB seen from the region $\mathcal{D}_2 - R_2$ has become $-\frac{\pi}{2}\gamma_2 + \eta_2 + 2\pi$. The point $\overline{P_2}$ is defined to be the one in the middle of the arc AB in $\mathcal{D}_2 - R_2$. The region $\mathcal{D}_2 - R_2$ is further mapped by $\widehat{w} = e^{i\theta}\widehat{v}^{\beta}$ into the region E_2 in the $\widehat{w}$-plane, in such a way that $\overline{P_2}$ is now taken to the point e^{0i}. The set of data drawn from the above consideration is:

$$\text{The angle of the sector } E_2 : (2\pi - \pi\gamma_2)\beta\,, \tag{3.16}$$

$$\text{The angle of the line segment } QA : \left(\frac{\pi}{2}\gamma_2 + \eta_2\right)\beta + \theta \in [-\pi, 0]\,, \tag{3.17}$$

$$\text{The angle of the line segment } QB : \left(-\frac{\pi}{2}\gamma_2 + \eta_2 + 2\pi\right)\beta + \theta \in [0, \pi]\,, \tag{3.18}$$

$$\begin{aligned}\text{Equality of two angles } \angle AQ\overline{P_2}, \angle BQ\overline{P_2} : & -\left(\frac{\pi}{2}\gamma_2 + \eta_2\right)\beta - \theta = \\ & = \left(-\frac{\pi}{2}\gamma_2 + \eta_2 + 2\pi\right)\beta + \theta\,.\end{aligned} \tag{3.19}$$

Finally, for the eq. (3.9) to be true, the exponents of h in both sides must be identical. Since

$$\begin{aligned} F_1 \circ f(z) &= h^{-1}\left(e^{i(\phi + \eta_1\alpha)} h(z)^{\gamma_1\alpha}\right), \\ \widehat{F}_2 \circ g \circ I(z) &= h^{-1}\left(e^{i(\theta + \eta_2\beta)} h(I(z))^{\gamma_2\beta}\right), \end{aligned} \tag{3.20}$$

this condition gives

$$\gamma_1 \alpha = \gamma_2 \beta \,. \tag{3.21}$$

We have now finished gathering the pieces needed to determine the values of $\alpha, \beta, \phi, \theta$, so we begin to solve them. In order for two regions E_1 and E_2 to be precisely sewn together and to form a unit disk, their angles given in (3.12) and (3.16) must add up to 2π,

$$(2\pi - \pi\gamma_1)\alpha + (2\pi - \pi\gamma_2)\beta = 2\pi \,. \tag{3.22}$$

Solving (3.22) and (3.21) simultaneously, we find

$$\alpha = \frac{\gamma_2}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2} \,, \qquad \beta = \frac{\gamma_1}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2} \,. \tag{3.23}$$

Then eqs. (3.15) and (3.19) give

$$\phi = -\pi + \frac{(\pi - \eta_1)\gamma_2}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2} \,, \qquad \theta = -\frac{(\pi + \eta_2)\gamma_1}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2} \,. \tag{3.24}$$

Since the above construction guarantees that the line segments QA, QB in E_1 should overlap on the $\widehat{w}$-plane with the corresponding ones QA, QB in E_2, their angles must agree for consistency: (3.13)=(3.17), (3.18)=(3.14) $+2\pi$. One can verify that these two equations indeed hold with the choices (3.23) and (3.24). Finally we must also examine whether $F_1 \circ f(z) = \widehat{F}_2 \circ g \circ I(z)$ is satisfied. From eqs. (3.20), (3.23) and (3.24), we have

$$F_1 \circ f(z) \,=\, h^{-1}\left[e^{i\pi \frac{\gamma_1(\gamma_2 - 1)}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2}} h(z)^{\frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2}} \right] , \tag{3.25}$$

$$\widehat{F}_2 \circ g \circ I(z) \,=\, h^{-1}\left[e^{-i\pi \frac{\gamma_1}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2}} h(I(z))^{\frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2}} \right] . \tag{3.26}$$

In general, the expression (3.26) is not well-defined because the discontinuous transformation $I(z)$ gives rise to a factor of -1 through the relation $h(I(z)) = -h(z)$. Throughout this paper we precisely define it as

$$h(I(z)) = -h(z) \equiv e^{\pi i} h(z) \,. \tag{3.27}$$

By this definition, it can easily be seen that $\widehat{F}_2 \circ g \circ I(z)$ exactly agrees with eq. (3.25), as desired.

Substituting (3.23) and (3.24) into (3.11), we have reached the following general formula

$$\begin{aligned} F_1(z) \,&=\, h^{-1}\left(\exp\left[i\left((2n+1)\pi + \frac{(\pi - \eta_1)\gamma_2}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2} \right) \right] h(z)^{\frac{\gamma_2}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2}} \right) , \\ \widehat{F}_2(z) \,&=\, h^{-1}\left(\exp\left[i\left((2n+2)\pi - \frac{(\pi + \eta_2)\gamma_1}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2} \right) \right] h(z)^{\frac{\gamma_1}{\gamma_1 + \gamma_2 - \gamma_1\gamma_2}} \right) , \end{aligned} \tag{3.28}$$

which solves the problem of finding functions satisfying

$$F_1 \circ f(z) = \widehat{F}_2 \circ g \circ I(z)$$

for f, g of the special form (3.10). In writing (3.28), we have added $2(n+1)\pi$ commonly to ϕ and θ to make the formula more flexible.

4. On some technical details

4.1 Does $|\mathcal{I}\rangle$ act as the identity element?

To begin with, we want to show that we can prove

$$\langle \phi, \mathcal{I} * \psi \rangle = \langle \phi, \psi * \mathcal{I} \rangle = \langle \phi, \psi \rangle \tag{4.1}$$

from the definition (2.8) of $\mathcal{I}$ and the algebraic statement (3.8) of the gluing theorem. As mentioned in section 2, we take $\langle \phi|$ and $|\psi\rangle$ to be Fock space states. Let us consider the most left-hand side of eq. (4.1). Inserting the complete set of basis $\sum |\Phi_r\rangle\langle \Phi_r^c|$, we get

$$\begin{aligned} \langle \phi, \mathcal{I} * \psi \rangle &= \sum_r \langle \phi, \Phi_r * \psi \rangle \langle \Phi_r^c, \mathcal{I} \rangle \\ &= \sum_r \left\langle f_1^{(3)} \circ \phi(0)\ f_2^{(3)} \circ \Phi_r(0)\ f_3^{(3)} \circ \psi(0) \right\rangle \langle f_{\mathcal{I}} \circ \Phi_r^c(0) \rangle \,, \end{aligned} \tag{4.2}$$

where use was made of definition (2.2) of the 3-string vertex and that (2.8) of $\mathcal{I}$.[5] From here on we omit the subscript "UHP" of the correlators for simplicity, as long as no misunderstanding could occur. Summing over r with the help of the gluing theorem (3.8) gives

$$\langle \phi, \mathcal{I} * \psi \rangle = \left\langle F_1 \circ f_3^{(3)} \circ \psi(0)\ F_1 \circ f_1^{(3)} \circ \phi(0) \right\rangle , \tag{4.3}$$

where F_1 must satisfy

$$F_1 \circ f_2^{(3)}(z) = \widehat{F}_2 \circ f_{\mathcal{I}} \circ I(z) \tag{4.4}$$

for $f_2^{(3)}(z)$ given in (2.5) and $f_{\mathcal{I}}(z)$ in (2.9). We can find such functions

$$\begin{aligned} F_1(z) &= h^{-1}\left(e^{2(n+1)\pi i + \frac{1}{2}\pi i} h(z)^{3/2} \right) , \\ \widehat{F}_2(z) &= h^{-1}\left(e^{2(n+1)\pi i - \frac{1}{2}\pi i} h(z)^{1/2} \right) , \end{aligned} \tag{4.5}$$

by applying the general formula (3.28) to this case ($\eta_1 = \eta_2 = 0, \gamma_1 = 2/3, \gamma_2 = 2$). Then, we can compute

$$\begin{aligned} F_1 \circ f_3^{(3)}(z) &= h^{-1}\left(e^{(2n+2)\pi i + \frac{1}{2}\pi i} \left(e^{-4\pi i/3} h(z)^{2/3} \right)^{3/2} \right) \\ &= h^{-1}\left(e^{\left(2n+\frac{1}{2}\right)\pi i} h(z) \right) \equiv h^{-1} \circ R_{\pi/2} \circ h(z) = \mathcal{R}_{\pi/2}(z) \,, \end{aligned} \tag{4.6}$$

where we have defined R_θ to be the rotation of the complex plane by an angle θ, namely $R_\theta(z) = e^{i\theta} z$, and $\mathcal{R}_\theta(z) = h^{-1} \circ R_\theta \circ h(z)$ which belongs to $\mathrm{SL}(2,\mathbb{R})$. We have used $f_3^{(3)}(z) = h^{-1}(e^{-4\pi i/3} h(z)^{2/3})$ instead of (2.6) because the map $F_1 \circ h^{-1}$ has been defined on the sector $\mathcal{D}_1 - R_1 = \{z; |z| \le 1, -\frac{5}{3}\pi \le \arg z \le -\pi/3\}$, as discussed in detail in the last section. If we notice the following relation

$$h \circ I(z) = h\left(-\frac{1}{z}\right) = \frac{1 - i/z}{1 + i/z} = -\frac{1 + iz}{1 - iz} = -h(z) = e^{\pi i} h(z) \,, \tag{4.7}$$

[5] Both the symmetric property of the BPZ inner product, $\langle A, B \rangle = \langle I \circ A(0)\, B(0) \rangle = \langle B, A \rangle$, and the cyclic symmetry of the 3-string vertex immediately follow from the $\mathrm{SL}(2,\mathbb{R})$-invariance of the correlation functions.

we have

$$F_1 \circ f_1^{(3)}(z) = h^{-1}\left(e^{2(n+1)\pi i - \frac{1}{2}\pi i} h(z)\right) = h^{-1}\left(e^{2n\pi i + \frac{1}{2}\pi i} h \circ I(z)\right) = \mathcal{R}_{\pi/2} \circ I(z)\,. \quad (4.8)$$

Substituting (4.6) and (4.8) into (4.3), we finally obtain[6]

$$\langle \phi, \mathcal{I} * \psi \rangle = \langle \mathcal{R}_{\pi/2} \circ \psi(0)\ \mathcal{R}_{\pi/2} \circ I \circ \phi(0) \rangle = \langle I \circ \phi(0)\ \psi(0) \rangle = \langle \phi, \psi \rangle\,, \quad (4.9)$$

which follows from the invariance of the correlator under the $\mathrm{SL}(2,\mathbb{R})$-map $\mathcal{R}_{\pi/2}$. In exactly the same way, we can also prove $\langle \phi, \psi * \mathcal{I} \rangle = \langle \phi, \psi \rangle$.

Furthermore, we can show

$$\langle \phi, \mathcal{I} * \mathcal{O}\mathcal{I} \rangle = \langle \phi, \mathcal{O}\mathcal{I} * \mathcal{I} \rangle = \langle \phi, \mathcal{O}\mathcal{I} \rangle\,, \quad (4.10)$$

where $\mathcal{O}$ denotes a local vertex operator acting on $|\mathcal{I}\rangle$, or a contour-integrate of such an operator. This relation will play an important rôle in section 5. Making use of the gluing theorem twice, we get

$$\begin{aligned}
\langle \phi, \mathcal{I} * \mathcal{O}\mathcal{I} \rangle &= \sum_{r,s} \langle \phi, \Phi_r * \Phi_s \rangle \langle \Phi_r^c, \mathcal{I} \rangle \langle \Phi_s^c, \mathcal{O}\mathcal{I} \rangle \\
&= \sum_{r,s} \left\langle f_1^{(3)} \circ \phi(0)\ f_2^{(3)} \circ \Phi_r(0)\ f_3^{(3)} \circ \Phi_s(0) \right\rangle \langle f_{\mathcal{I}} \circ \Phi_r^c(0) \rangle \times \\
&\qquad \times \langle f_{\mathcal{I}} \circ I \circ \mathcal{O}\ f_{\mathcal{I}} \circ \Phi_s^c(0) \rangle \\
&= \sum_{s} \left\langle F_1 \circ f_3^{(3)} \circ \Phi_s(0)\ F_1 \circ f_1^{(3)} \circ \phi(0) \right\rangle \langle f_{\mathcal{I}} \circ I \circ \mathcal{O}\ f_{\mathcal{I}} \circ \Phi_s^c(0) \rangle \\
&= \left\langle G_1 \circ F_1 \circ f_1^{(3)} \circ \phi(0)\ \widehat{G}_2 \circ f_{\mathcal{I}} \circ I \circ \mathcal{O} \right\rangle, \qquad (4.11)
\end{aligned}$$

where

$$F_1 \circ f_2^{(3)}(z) = \widehat{F}_2 \circ f_{\mathcal{I}} \circ I(z)\,, \quad (4.12)$$

$$G_1 \circ \left(F_1 \circ f_3^{(3)}\right)(z) = \widehat{G}_2 \circ f_{\mathcal{I}} \circ I(z) \quad (4.13)$$

must be satisfied. Since eq. (4.12) is the same as eq. (4.4), the answer should be given by (4.5). The general formula (3.28) for $f(z) = F_1 \circ f_3^{(3)}(z) = h^{-1}\left(e^{2n\pi i + \frac{1}{2}\pi i} h(z)\right)$ (eq. (4.6)) and $g = f_{\mathcal{I}}$ gives us

$$\begin{aligned}
G_1(z) &= h^{-1}\left(e^{2\pi i(1+n'-2n)} h(z)^2\right), \\
\widehat{G}_2(z) &= h^{-1}\left(e^{\pi i(2n'+1)} h(z)\right), \qquad (4.14)
\end{aligned}$$

with n' some integer. From the relations

$$\begin{aligned}
G_1 \circ F_1 \circ f_1^{(3)}(z) &= h^{-1}\left(e^{2n'\pi i + \pi i} h(z)^2\right) = \mathcal{R}_\pi \circ f_{\mathcal{I}}(z)\,, \\
\widehat{G}_2 \circ f_{\mathcal{I}} \circ I(z) &= h^{-1}\left(e^{2(n'+1)\pi i + \pi i} h(z)^2\right) = \mathcal{R}_\pi \circ f_{\mathcal{I}}(z)\,,
\end{aligned}$$

[6]Note that in order for $\langle \phi, \psi \rangle$ to have a non-vanishing value ϕ and ψ must have the opposite Grassmannality from each other.

it follows that

$$\langle \phi, \mathcal{I} * \mathcal{O}\mathcal{I} \rangle = \langle f_{\mathcal{I}} \circ \phi(0) \; f_{\mathcal{I}} \circ \mathcal{O} \rangle \, . \tag{4.15}$$

On the other hand, the evaluation of $\langle \phi, \mathcal{O}\mathcal{I} \rangle$ leads to

$$\langle \phi, \mathcal{O}\mathcal{I} \rangle = \langle \mathcal{I}, (I \circ \mathcal{O})\phi \rangle = \langle f_{\mathcal{I}} \circ I \circ \mathcal{O} \; f_{\mathcal{I}} \circ \phi(0) \rangle = \langle f_{\mathcal{I}} \circ \phi(0) \; f_{\mathcal{I}} \circ \mathcal{O} \rangle \tag{4.16}$$

because

$$f_{\mathcal{I}} \circ I(z) = h^{-1}\left(h(I(z))^2\right) = h^{-1}\left((-h(z))^2\right) = h^{-1}(h(z)^2) = f_{\mathcal{I}}(z) \, . \tag{4.17}$$

Comparing (4.15) with (4.16), we conclude that

$$\langle \phi, \mathcal{I} * \mathcal{O}\mathcal{I} \rangle = \langle \phi, \mathcal{O}\mathcal{I} \rangle \, , \tag{4.18}$$

as stated earlier. The remaining side in eq. (4.10) can be proven in a similar manner.

Finally we want to evaluate

$$\langle \phi, \mathcal{O}_A \mathcal{I} * \mathcal{O}_B \mathcal{I} \rangle \, . \tag{4.19}$$

Actually we do not need to calculate any more. One can find

$$\langle \phi, \mathcal{O}_A \mathcal{I} * \mathcal{O}_B \mathcal{I} \rangle = \langle a_1 \circ \phi(0) \; a_2 \circ \mathcal{O}_A \; a_3 \circ \mathcal{O}_B \rangle \, , \tag{4.20}$$

with

$$\begin{aligned}
a_1(z) &= G_1 \circ F_1 \circ f_1^{(3)}(z) = h^{-1}\left(e^{2n'\pi i + \pi i} h(z)^2\right) = \mathcal{R}_\pi \circ f_{\mathcal{I}}(z) \, , \\
a_2(z) &= G_1 \circ \widehat{F}_2 \circ f_{\mathcal{I}} \circ I(z) = h^{-1}\left(e^{2\pi i(n'+1) + \pi i} h(z)^2\right) = \mathcal{R}_\pi \circ f_{\mathcal{I}}(z) \, , \\
a_3(z) &= \widehat{G}_2 \circ f_{\mathcal{I}} \circ I(z) = h^{-1}\left(e^{2(n'+1)\pi i + \pi i} h(z)^2\right) = \mathcal{R}_\pi \circ f_{\mathcal{I}}(z) \, ,
\end{aligned}$$

where $F_1, \widehat{F}_2, G_1, \widehat{G}_2$ have already been given in eqs. (4.5) and (4.14). Now consider the case

$$\mathcal{O}_A = \mathcal{O}_B = \mathcal{Q}^\epsilon = \frac{1}{2i}\left(e^{-i\epsilon} c(ie^{i\epsilon}) - e^{i\epsilon} c(-ie^{-i\epsilon})\right) , \tag{4.21}$$

which will appear in section 5. In this case eq. (4.20) reduces to

$$\begin{aligned}
\langle \phi, \mathcal{Q}^\epsilon \mathcal{I} * \mathcal{Q}^\epsilon \mathcal{I} \rangle &= -\frac{1}{4} \left\langle a_1 \circ \phi(0) \left(\frac{e^{-i\epsilon}}{a_2'(ie^{i\epsilon})} c(a_2(ie^{i\epsilon})) - \frac{e^{i\epsilon}}{a_2'(-ie^{-i\epsilon})} c(a_2(-ie^{-i\epsilon})) \right) \times \right. \\
&\qquad \left. \times \left(\frac{e^{-i\epsilon}}{a_3'(ie^{i\epsilon})} c(a_3(ie^{i\epsilon})) - \frac{e^{i\epsilon}}{a_3'(-ie^{-i\epsilon})} c(a_3(-ie^{-i\epsilon})) \right) \right\rangle .
\end{aligned} \tag{4.22}$$

The fact $a_2(z) = a_3(z)$ means that the two *local fermionic* operators are inserted at the same point, which makes the above correlator vanish. One may think that the conformal factors could provide infinities, but it does not happen as long as ϵ is kept finite. Hence we have found that

$$\langle \phi, \mathcal{Q}^\epsilon \mathcal{I} * \mathcal{Q}^\epsilon \mathcal{I} \rangle = 0 \, , \qquad \text{for arbitrary non-zero } \epsilon \, , \tag{4.23}$$

irrespective of the details of the state $|\phi\rangle$.

Here we give a brief comment on the use of the operator formulation of string field theory [22, 23]. The expression for the identity state there is given by [23]

$$|\mathcal{I}\rangle = \frac{1}{4i}b^+\left(\frac{\pi}{2}\right)b^-\left(\frac{\pi}{2}\right)\exp\left[\sum_{n=1}^{\infty}(-1)^n\left(-\frac{1}{2n}\alpha_{-n}\cdot\alpha_{-n}+c_{-n}b_{-n}\right)\right]c_0c_1|0\rangle\,. \tag{4.24}$$

The $*$-product of string field theory is expressed as

$$|A*B\rangle_3 = {}_1\langle A|_2\langle B||V_3\rangle_{123}\,,$$

where the 3-string vertex $|V_3\rangle$ is defined in terms of Neumann coefficients U^{rs}_{nm}, X^{rs}_{nm} as

$$|V_3\rangle_{123} = \exp\left[-\sum_{r,s=1}^{3}\left(\frac{1}{2}\sum_{n,m\geq 0}a_n^{(r)\dagger}\cdot U^{rs}_{nm}a_m^{(s)\dagger}+\sum_{n\geq 1,m\geq 0}c_{-n}^{(r)}X^{rs}_{nm}b_{-m}^{(s)}\right)\right]|+\rangle_{123}\,.$$

The BPZ conjugation is implemented by the reflector state

$$\begin{aligned}{}_{12}\langle R| &= {}_{12}\langle\tilde{+}|\exp\left[-\sum_{n=0}^{\infty}(-1)^n a_n^{(1)}\cdot a_n^{(2)}-\sum_{n=1}^{\infty}(-1)^n\left(c_n^{(1)}b_n^{(2)}+c_n^{(2)}b_n^{(1)}\right)\right]\left(c_0^{(1)}+c_0^{(2)}\right),\\ |R\rangle_{12} &= \left(b_0^{(1)}-b_0^{(2)}\right)\exp\left[-\sum_{n=0}^{\infty}(-1)^n a_n^{(1)\dagger}\cdot a_n^{(2)\dagger}+\sum_{n=1}^{\infty}(-1)^n\left(c_{-n}^{(1)}b_{-n}^{(2)}+c_{-n}^{(2)}b_{-n}^{(1)}\right)\right]|+\rangle_{12}\,.\end{aligned}$$

Then, the condition that $\mathcal{I}*A=A$ holds for any state A is equivalent to the statement

$${}_1\langle\mathcal{I}||V_3\rangle_{123} = |R\rangle_{23}\,. \tag{4.25}$$

Although one can show that this indeed holds true in the matter sector up to an overall determinant factor, explicit calculations of ${}_1\langle\mathcal{I}_{\rm g}||V_3^{\rm g}\rangle_{123}$ in the ghost sector give us an expression which looks very different from the reflector [9]. To make matters worse, naïve computations[7] have led us to a strangely-looking result: $\mathcal{I}_{\rm g}*\mathcal{I}_{\rm g}=0$. Of course, since this expression is multiplied by a determinant of infinite matrices which might be divergent, it may be possible that the infinite determinant factor compensates for the apparent vanishing of $\mathcal{I}_{\rm g}*\mathcal{I}_{\rm g}$. Even if this would be the case, however, there seems to be no natural way of regularizing it. For these depressing results, we have given up dealing with the identity state in the operator formalism.

4.2 Concerning the BRST operator

We will later give arguments which depend on the special properties possessed by the BRST charge Q_B appearing in the original action (5.1). In this subsection we prepare for them.

We first fix the possible ambiguity of the BRST current $j_B(z)$ by requiring that it be a primary field of conformal weight 1. The result is

$$j_B = cT^{\rm m} + :bc\partial c: + \frac{3}{2}\partial^2 c\,, \tag{4.26}$$

where $T^{\rm m}$ denotes the matter part of the energy-momentum tensor. Since the OPE of j_B with itself has the single pole proportional to $c^{\rm m}-26$ with $c^{\rm m}$ denoting the matter central

[7] We used the relations among infinite matrices formally.

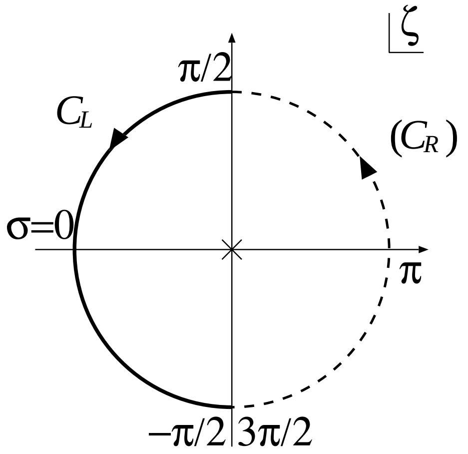

Figure 7: The integration contour C_L for Q_L.

charge, it immediately follows that in the case of critical bosonic string theory

$$\{Q_B, j_B(z)\} = 0\,, \qquad \text{for any } z\,. \tag{4.27}$$

Here the BRST charge Q_B has been defined by

$$Q_B = \oint_C \frac{d\zeta}{2\pi i} j_B(\zeta) = -\int_0^\pi \frac{d\sigma}{2\pi}(j_B(\sigma) + \widetilde{\jmath}_B(\sigma))\,, \tag{4.28}$$

where the integration contour C encircles z counterclockwise, and we have used the convention that $j_B(\zeta)$ is a holomorphic field defined over the whole complex plane via the doubling trick, whereas $j_B(\sigma), \widetilde{\jmath}_B(\sigma)$ are holomorphic and antiholomorphic fields, respectively, defined only on the upper half-plane. As a result, we in particular obtain

$$\{Q_B, Q_L\} = 0\,, \tag{4.29}$$

where

$$Q_L \equiv -\int_0^{\pi/2} \frac{d\sigma}{2\pi}(j_B(\sigma) + \widetilde{\jmath}_B(\sigma)) = \int_{C_L} \frac{d\zeta}{2\pi i} j_B(\zeta)\,, \tag{4.30}$$

and the contour C_L is indicated in figure 7[8] for Q_L acting on $|\phi\rangle = \phi(0)|0\rangle$. We further define Q_R by $Q_B - Q_L$.

Since Q_B has been defined to be the integral of a primary field of conformal weight 1, it commutes with the conformal transformation. This fact can also be seen from $[Q_B, L_n^{\text{tot}}] = 0$ for any n. It then follows that the BRST charge annihilates arbitrary wedge states $\langle n|$ defined in (2.10) and (2.11), because

$$\begin{aligned}\langle n|Q_B|\phi\rangle &= \left\langle f^{(n)} \circ (Q_B\phi)(0)\right\rangle = \left\langle Q_B \left(f^{(n)} \circ \phi(0)\right)\right\rangle \\ &= \left\langle \oint_C \frac{d\zeta}{2\pi i} j_B(\zeta) \left(f^{(n)} \circ \phi(0)\right)\right\rangle = 0 \end{aligned} \tag{4.31}$$

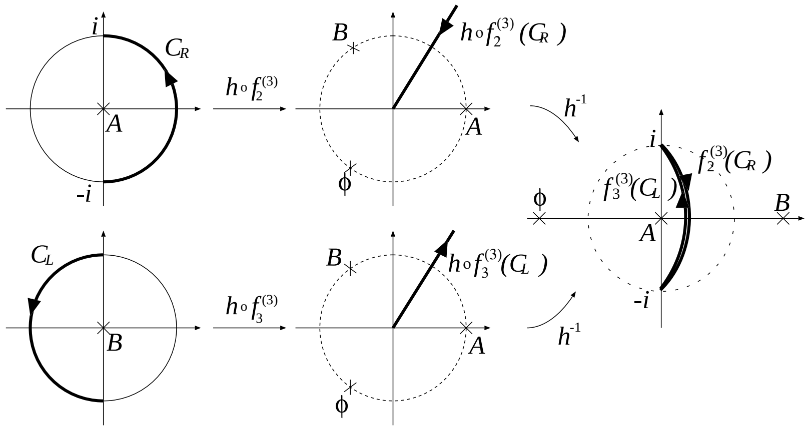

Figure 8: The mapping of contours C_R, C_L by $f_2^{(3)}, f_3^{(3)}$.

by deforming the contour C until it shrinks to a point at infinity where there is no other operator. In particular, we have $\langle \mathcal{I}|Q_B = 0$.

An important rôle is played in later calculations by a sort of "partial integration formula"

$$\langle \phi, (Q_R A) * B\rangle = -(-1)^{|A|} \langle \phi, A * (Q_L B)\rangle \,, \tag{4.32}$$

where $|A|$ denotes the Grassmannality of A. When both A and B are Fock space states, it can be shown simply by considering the integration contours. The contours C_R and C_L are mapped under $f_2^{(3)}$ and $f_3^{(3)}$, respectively, as in figure 8. Then

$$\begin{aligned}
&\langle \phi, (Q_R A) * B + (-1)^{|A|} A * (Q_L B)\rangle = \\
&= \left\langle f_1^{(3)} \circ \phi(0) \int_{f_2^{(3)}(C_R)} \frac{dz}{2\pi i} j_B(z)\ f_2^{(3)} \circ A(0)\ f_3^{(3)} \circ B(0) \right\rangle + \\
&\quad + \left\langle f_1^{(3)} \circ \phi(0) \int_{f_3^{(3)}(C_L)} \frac{dz}{2\pi i} j_B(z)\ f_2^{(3)} \circ A(0)\ f_3^{(3)} \circ B(0) \right\rangle \\
&= \left\langle f_1^{(3)} \circ \phi(0) \int_{f_2^{(3)}(C_R)+f_3^{(3)}(C_L)} \frac{dz}{2\pi i} j_B(z)\ f_2^{(3)} \circ A(0)\ f_3^{(3)} \circ B(0) \right\rangle .
\end{aligned} \tag{4.33}$$

Since the contour $C = f_2^{(3)}(C_R) + f_3^{(3)}(C_L)$ is closed and does not encircle any operator inside it, the above expression (4.33) vanishes due to the holomorphicity of $j_B(z)$. Hence we have shown eq. (4.32).

Even when A and B are of the form

$$|A\rangle = \mathcal{O}_A|m\rangle \,, \qquad |B\rangle = \mathcal{O}_B|m\rangle \,, \tag{4.34}$$

where $\mathcal{O}_A, \mathcal{O}_B$ represent local operators or contour-integrates of them, and $|m\rangle$ is an arbitrary wedge state, we can also give a similar argument to the above one. Using the gluing

[8] We are following the convention that two world-sheet coordinates (σ, τ) and $(z, \bar{z})$ are related by $z = -e^{-i\sigma+\tau}$, $\bar{z} = -e^{i\sigma+\tau}$.

theorem twice, we have obtained

$$\Big\langle \phi, (Q_R\mathcal{O}_A|m\rangle) * \mathcal{O}_B|m\rangle + (-1)^{|\mathcal{O}_A|}\mathcal{O}_A|m\rangle * (Q_L\mathcal{O}_B|m\rangle)\Big\rangle = \tag{4.35}$$
$$= \Big\langle G_1\circ F_1\circ f_1^{(3)}\circ\phi(0)\ \int_{\mathcal{C}}\frac{dz}{2\pi i}j_B(z)\ G_1\circ\widehat{F}_2\circ f^{(m)}\circ I\circ\mathcal{O}_A\widehat{G}_2\circ f^{(m)}\circ I\circ\mathcal{O}_B\Big\rangle\,,$$

where

$$\mathcal{C} = G_1\circ\widehat{F}_2\circ f^{(m)}\circ I(C_R) + \widehat{G}_2\circ f^{(m)}\circ I(C_L) \tag{4.36}$$

and $F_1, \widehat{F}_2, G_1, \widehat{G}_2$ satisfy

$$F_1\circ f_2^{(3)}(z) = \widehat{F}_2\circ f^{(m)}\circ I(z)\,,\qquad G_1\circ\left(F_1\circ f_3^{(3)}\right)(z) = \widehat{G}_2\circ f^{(m)}\circ I(z)\,, \tag{4.37}$$

so these are explicitly given by

$$F_1(z) = h^{-1}\left(e^{i(2n+1)\pi+i\pi\frac{3}{m+1}}h(z)^{\frac{3}{m+1}}\right), \tag{4.38}$$
$$\widehat{F}_2(z) = h^{-1}\left(e^{2(n+1)\pi i-i\pi\frac{m}{m+1}}h(z)^{\frac{m}{m+1}}\right), \tag{4.39}$$
$$G_1(z) = h^{-1}\left(e^{i(2n'+1)\pi-2n\pi i\frac{m+1}{2m-1}+\pi i\frac{1}{2m-1}}h(z)^{\frac{m+1}{2m-1}}\right), \tag{4.40}$$
$$\widehat{G}_2(z) = h^{-1}\left(e^{2(n'+1)\pi i-i\pi\frac{m}{2m-1}}h(z)^{\frac{m}{2m-1}}\right), \tag{4.41}$$

with n, n' integers. We note that

$$\begin{aligned} F_1\circ f_3^{(3)}(z) &= h^{-1}\left(e^{i(2n+1)\pi+i\pi\frac{3}{m+1}}\left(e^{-\frac{4\pi i}{3}}h(z)^{2/3}\right)^{\frac{3}{m+1}}\right) \\ &= h^{-1}\left(e^{2n\pi i+i\pi\frac{m}{m+1}}h(z)^{\frac{2}{m+1}}\right). \end{aligned} \tag{4.42}$$

According to the arguments given in subsection 3.2, these conformal maps are defined over the regions

$$F_1\circ h^{-1}\ :\ \left\{z; -\frac{5}{3}\pi \le \arg z \le -\frac{\pi}{3}\right\}, \tag{4.43}$$
$$\widehat{F}_2\circ h^{-1}\ :\ \left\{z; \frac{\pi}{m} \le \arg z \le \frac{2m-1}{m}\pi\right\}, \tag{4.44}$$
$$G_1\circ h^{-1}\ :\ \left\{z; (2n-1)\pi \le \arg z \le 2n\pi + \frac{m-1}{m+1}\pi\right\}, \tag{4.45}$$
$$\widehat{G}_2\circ h^{-1}\ :\ \left\{z; \frac{\pi}{m} \le \arg z \le \frac{2m-1}{m}\pi\right\}, \tag{4.46}$$

respectively, where we have not restricted them to the inside $|z|\le 1$ of the unit disk because we are considering the full complex plane due to the doubling trick.

Now let us see where the integration contours C_R, C_L are mapped under $G_1\circ\widehat{F}_2\circ f^{(m)}\circ I$ and $\widehat{G}_2\circ f^{(m)}\circ I$, respectively. The contour C_R is first mapped by the inversion I to the left semi-circle (figure 9*b*), which is subsequently mapped under $h\circ f^{(m)}$ into the semi-infinite straight line whose angle is $-\pi/m$ to the positive real axis (figure 9*c*). Since this line lies outside the range (4.44) of definition of the map $\widehat{F}_2\circ h^{-1}$, we redefine its angle as

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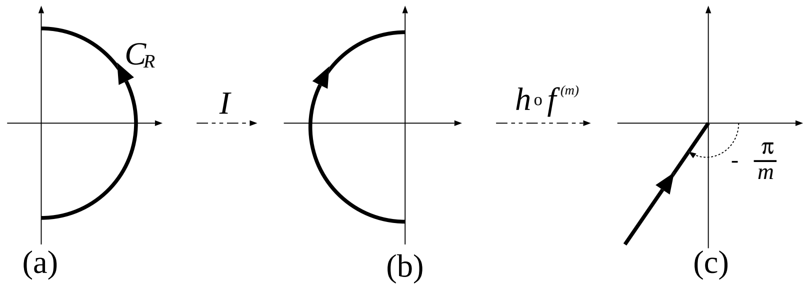

Figure 9: The mappings of C_R.

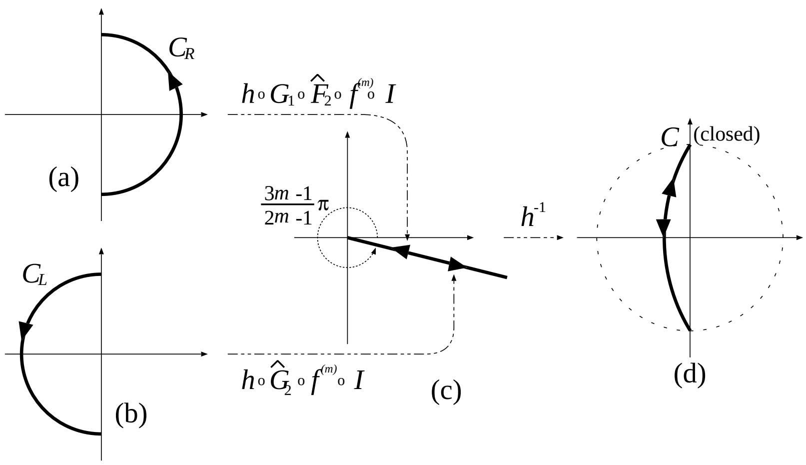

Figure 10: The contour $\mathcal{C} = G_1 \circ \widehat{F}_2 \circ f^{(m)} \circ I(C_R) + \widehat{G}_2 \circ f^{(m)} \circ I(C_L)$.

$-\pi/m + 2\pi = \pi(2m-1)/m$, which is contained in the domain (4.44). This line is mapped under $h \circ \widehat{F}_2 \circ h^{-1}$ (4.39) to another semi-infinite straight line with an angle

$$\frac{2m-1}{m}\pi \times \frac{m}{m+1} + 2(n+1)\pi - \pi\frac{m}{m+1} = 2n\pi + 2\pi + \frac{m-1}{m+1}\pi\,. \tag{4.47}$$

This angle, however, is outside the range (4.45) of definition of $G_1 \circ h^{-1}$, so that it should be redefined to be $2n\pi + \frac{m-1}{m+1}\pi$. This line is finally mapped by $h \circ G_1 \circ h^{-1}$ (4.40) to a semi-infinite straight line of an angle

$$\begin{aligned}\left(2n\pi + \frac{m-1}{m+1}\pi\right) \times \frac{m+1}{2m-1} + (2n'+1)\pi - 2n\pi\frac{m+1}{2m-1} + \frac{\pi}{2m-1} &= \\ = 2n'\pi + \frac{3m-1}{2m-1}\pi\,. & \end{aligned} \tag{4.48}$$

Note that an arbitrary integer n has dropped out, and that the remaining one n' is clearly irrelevant. To summarize, the contour C_R is mapped by $h \circ G_1 \circ \widehat{F}_2 \circ f^{(m)} \circ I$ to the semi-infinite straight line of an angle $\frac{3m-1}{2m-1}\pi$ to the positive real axis, as indicated in figure 10c.

In a similar way, we can examine where C_L is mapped under $h \circ \widehat{G}_2 \circ f^{(m)} \circ I$ as

$$C_L \xrightarrow{\;I\;} -C_R \xrightarrow{h\circ f^{(m)}} \left\{z; \arg z = \frac{\pi}{m}\right\} \xrightarrow{h\circ\widehat{G}_2\circ h^{-1}}$$

$$\xrightarrow{h\circ\widehat{G}_2\circ h^{-1}} \left\{z;\arg z = 2n'\pi + \frac{3m-1}{2m-1}\pi\right\}, \tag{4.49}$$

which precisely coincides with $\{h\circ G_1\circ \widehat{F}_2\circ f^{(m)}\circ I(C_R)\}$ found earlier in (4.48). Moreover, we can easily show that the directions of these two transformed contours are opposite to each other (figure 10c). Mapped by h^{-1}, the contour $\mathcal{C}$ (4.36) becomes a "closed" curve with no room for any operator insertion, as shown in figure 10d. Therefore, we have found that the integral of $j_B(z)$ along the contour $\mathcal{C}$ appearing in (4.35) vanishes, which means that we can carry out the partial integration

$$\langle\phi, (Q_R\mathcal{O}_A|m\rangle) * \mathcal{O}_B|m\rangle\rangle = -(-1)^{|\mathcal{O}_A|}\langle\phi, \mathcal{O}_A|m\rangle * (Q_L\mathcal{O}_B|m\rangle)\rangle \tag{4.50}$$

in the case (4.34) as well.

We will show that the formula (4.32) also holds for the following "mixed" cases,

$$|A\rangle = \mathcal{O}_A|m\rangle\,, \qquad |B\rangle = B(0)|0\rangle \tag{4.51}$$

and

$$|A\rangle = A(0)|0\rangle\,, \qquad |B\rangle = \mathcal{O}_B|m\rangle\,. \tag{4.52}$$

For the former case (4.51), we have

$$\begin{aligned}\Big\langle\phi, (Q_R\mathcal{O}_A|m\rangle) * B + (-1)^{|\mathcal{O}_A|}\mathcal{O}_A|m\rangle * (Q_LB)\Big\rangle &= \\ = \Big\langle F_1\circ f_1^{(3)}\circ\phi(0)\int_{\mathcal{C}^\ell}\frac{dz}{2\pi i}j_B(z)\ \widehat{F}_2\circ f^{(m)}\circ I\circ\mathcal{O}_A\ F_1\circ f_3^{(3)}\circ B(0)\Big\rangle&, \end{aligned}\tag{4.53}$$

where F_1 and $\widehat{F}_2$ have been given in eqs. (4.38)–(4.39), and

$$\mathcal{C}^\ell = \widehat{F}_2\circ f^{(m)}\circ I(C_R) + F_1\circ f_3^{(3)}(C_L)\,. \tag{4.54}$$

In the same spirit as in eq. (4.49), we follow where the contours C_R and C_L are mapped to in (4.54) as

$$\begin{aligned} C_R \xrightarrow{\ I\ } -C_L \xrightarrow{h\circ f^{(m)}} \left\{z;\arg z = -\frac{\pi}{m}\cong\frac{2m-1}{m}\pi\right\} \xrightarrow{h\circ\widehat{F}_2\circ h^{-1}} \\ \xrightarrow{h\circ\widehat{F}_2\circ h^{-1}} \left\{z;\arg z = 2(n+1)\pi + \frac{m-1}{m+1}\pi\right\}, \end{aligned}\tag{4.55}$$

$$C_L \xrightarrow{h\circ f_3^{(3)}} \left\{z;\arg z = \frac{\pi}{3}\cong -\frac{5}{3}\pi\right\} \xrightarrow{h\circ F_1\circ h^{-1}} \left\{z;\arg z = 2n\pi + \frac{m-1}{m+1}\pi\right\}, \tag{4.56}$$

where the symbol $\cong$ means that we have added or subtracted 2π according to the ranges of definition of the subsequent maps. Since the SL$(2,\mathbb{C})$-map $h^{-1}(z)$ to be performed finally does not give rise to any deficit angle, the above two contours (4.55), (4.56) precisely overlap with each other in the opposite direction on the global complex plane, despite the fact that they differ by 2π in angle. Hence the contour $\mathcal{C}^\ell$ can shrink to zero-size without picking up any poles, making the correlator in the right-hand side of eq. (4.53) vanish. So we have proven

$$\langle\phi, (Q_R\mathcal{O}_A|m\rangle) * B\rangle = -(-1)^{|\mathcal{O}_A|}\langle\phi, \mathcal{O}_A|m\rangle * (Q_LB)\rangle\,. \tag{4.57}$$

For the latter case (4.52), we go through similar steps:

$$\begin{aligned}&\Big\langle\phi,(Q_RA)*\mathcal{O}_B|m\rangle+(-1)^{|A|}A*(Q_L\mathcal{O}_B|m\rangle)\Big\rangle=\\&\quad=\Big\langle F_1^r\circ f_1^{(3)}\circ\phi(0)\int_{\mathcal{C}^r}\frac{dz}{2\pi i}j_B(z)\ F_1^r\circ f_2^{(3)}\circ A(0)\ \widehat{F}_2^r\circ f^{(m)}\circ I\circ\mathcal{O}_B\Big\rangle\,,\end{aligned}\tag{4.58}$$

where

$$\begin{aligned}F_1^r(z)&=h^{-1}\left(e^{2n\pi i+i\pi\frac{m+2}{m+1}}h(z)^{\frac{3}{m+1}}\right); &\arg h(z)\in\left[-\pi,\frac{\pi}{3}\right],\\ \widehat{F}_2^r(z)&=h^{-1}\left(e^{2n\pi i+i\pi\frac{m+2}{m+1}}h(z)^{\frac{m}{m+1}}\right); &\arg h(z)\in\left[\frac{\pi}{m},\frac{2m-1}{m}\pi\right],\\ \mathcal{C}^r&=F_1^r\circ f_2^{(3)}(C_R)+\widehat{F}_2^r\circ f^{(m)}\circ I(C_L)\,.\end{aligned}\tag{4.59}$$

The mappings of the contours C_L, C_R are as follows:

$$C_R\ \xrightarrow{h\circ f_2^{(3)}}\ \left\{z;\arg z=\frac{\pi}{3}\right\}\ \xrightarrow{h\circ F_1^r\circ h^{-1}}\ \left\{z;\arg z=2n\pi+\frac{m+3}{m+1}\pi\right\},\tag{4.60}$$

$$C_L\ \xrightarrow{h\circ f^{(m)}\circ I}\ \left\{z;\arg z=\frac{\pi}{m}\right\}\ \xrightarrow{h\circ \widehat{F}_2^r\circ h^{-1}}\ \left\{z;\arg z=2n\pi+\frac{m+3}{m+1}\pi\right\}.\tag{4.61}$$

Therefore, one again finds $\mathcal{C}^r\sim\{\text{pt.}\}$, so that

$$\langle\phi,(Q_RA)*\mathcal{O}_B|m\rangle\rangle=-(-1)^{|A|}\langle\phi,A*(Q_L\mathcal{O}_B|m\rangle)\rangle\,.\tag{4.62}$$

We note here some relations which have been established and will be used later.

$$\langle\phi,\mathcal{O}\mathcal{I}*Q_L\mathcal{I}\rangle=-(-1)^{|\mathcal{O}|}\langle\phi,Q_R\mathcal{O}\mathcal{I}*\mathcal{I}\rangle\,,\tag{4.63}$$

$$\langle\phi,Q_R\mathcal{I}*\mathcal{O}\mathcal{I}\rangle=-\langle\phi,\mathcal{I}*Q_L\mathcal{O}\mathcal{I}\rangle\,,\tag{4.64}$$

$$\langle\phi,Q_R\mathcal{I}*\psi\rangle=-\langle\phi,\mathcal{I}*Q_L\psi\rangle\,,\tag{4.65}$$

$$\langle\phi,\psi*Q_L\mathcal{I}\rangle=-(-1)^{|\psi|}\langle\phi,Q_R\psi*\mathcal{I}\rangle\,.\tag{4.66}$$

4.3 Regularized $\mathcal{Q}$ as an inner derivation

Now we consider an "inner derivation" of the form

$$\mathcal{Q}^\epsilon\mathcal{I}*\psi-(-1)^{|\psi|}\psi*\mathcal{Q}^\epsilon\mathcal{I}\,,\tag{4.67}$$

where $\mathcal{Q}^\epsilon$ will be defined in (5.14). Taking the BPZ inner product with a Fock space state $|\phi\rangle$ and calculating it with the gluing theorem, we obtain

$$\begin{aligned}&\langle\phi,\mathcal{Q}^\epsilon\mathcal{I}*\psi\rangle-(-1)^{|\psi|}\langle\phi,\psi*\mathcal{Q}^\epsilon\mathcal{I}\rangle=\\&\quad=\Big\langle F_1\circ f_3^{(3)}\circ\psi(0)\ F_1\circ f_1^{(3)}\circ\phi(0)\ \widehat{F}_2\circ f_{\mathcal{I}}\circ I\circ\mathcal{Q}^\epsilon\Big\rangle-\\&\quad-(-1)^{|\psi|}\Big\langle F_3\circ f_1^{(3)}\circ\phi(0)\ F_3\circ f_2^{(3)}\circ\psi(0)\ \widehat{F}_4\circ f_{\mathcal{I}}\circ I\circ\mathcal{Q}^\epsilon\Big\rangle\,,\end{aligned}\tag{4.68}$$

where F's satisfy

$$F_1\circ f_2^{(3)}(z)=\widehat{F}_2\circ f_{\mathcal{I}}\circ I(z)\,,\qquad F_3\circ f_3^{(3)}(z)=\widehat{F}_4\circ f_{\mathcal{I}}\circ I(z)\,.\tag{4.69}$$

The solution to the above equations is found to be

$$F_1(z) = h^{-1}\left(e^{2(n+1)\pi i+\frac{1}{2}\pi i}h(z)^{3/2}\right), \qquad \widehat{F}_2(z) = h^{-1}\left(e^{2(n+1)\pi i-\frac{1}{2}\pi i}h(z)^{1/2}\right),$$
$$F_3(z) = h^{-1}\left(e^{(2n'+1)\pi i+\frac{1}{2}\pi i}h(z)^{3/2}\right), \qquad \widehat{F}_4(z) = h^{-1}\left(e^{2(n'+1)\pi i-\frac{1}{2}\pi i}h(z)^{1/2}\right).$$

Then the right-hand side of eq. (4.68) becomes

$$\begin{aligned}&\langle \mathcal{R}_{\pi/2}\circ\psi(0)\ \mathcal{R}_{\pi/2}\circ I\circ\phi(0)\ \mathcal{R}_{\pi/2}\circ\mathcal{Q}^\epsilon\rangle -\\ &-(-1)^{|\psi|}\langle \mathcal{R}_{\pi/2}\circ\phi(0)\ \mathcal{R}_{\pi/2}\circ I\circ\psi(0)\ \mathcal{R}_{\pi/2}\circ\mathcal{Q}^\epsilon\rangle =\\ &\quad = \langle I\circ\phi(0)\ (\mathcal{Q}^\epsilon - I\circ\mathcal{Q}^\epsilon)\psi(0)\rangle = \langle\phi|(\mathcal{Q}^\epsilon - I\circ\mathcal{Q}^\epsilon)|\psi\rangle\,, \end{aligned} \tag{4.70}$$

where $\mathcal{R}_\theta$ has been defined in (4.6). From the definition (5.14) of $\mathcal{Q}^\epsilon$, we can explicitly write down $\mathcal{Q}^\epsilon - I\circ\mathcal{Q}^\epsilon$ as

$$\begin{aligned}\mathcal{Q}^\epsilon - I\circ\mathcal{Q}^\epsilon &= \frac{1}{2i}\Big(e^{-i\epsilon}c(ie^{i\epsilon}) - e^{i\epsilon}c(-ie^{-i\epsilon}) - e^{-i\epsilon}I\circ c(ie^{i\epsilon}) + e^{i\epsilon}I\circ c(-ie^{-i\epsilon})\Big)\\ &= 2\times\frac{1}{4i}\Big(e^{-i\epsilon}c(ie^{i\epsilon}) + e^{i\epsilon}c(ie^{-i\epsilon}) - e^{-i\epsilon}c(-ie^{i\epsilon}) - e^{i\epsilon}c(-ie^{-i\epsilon})\Big)\\ &\equiv 2\mathcal{Q}^A_\epsilon\,. \end{aligned} \tag{4.71}$$

Making use of the operator $\mathcal{Q}^A_\epsilon$ defined above, which naïvely seems to approach $\mathcal{Q}/(g_o^2\kappa_0)^{1/3}$ $= \frac{1}{2i}(c(i)-c(-i))$ in the limit $\epsilon\to 0$ (see eq. (5.7)), the expression (4.67) can be written as

$$\frac{1}{2}\langle\phi,\mathcal{Q}^\epsilon\mathcal{I}*\psi\rangle - \frac{(-1)^{|\psi|}}{2}\langle\phi,\psi*\mathcal{Q}^\epsilon\mathcal{I}\rangle = \langle\phi,\mathcal{Q}^A_\epsilon\psi\rangle\,. \tag{4.72}$$

In fact, this relation has already been stated without proof in [11, eq.(2.27)]. (Our $\mathcal{Q}^A_\epsilon$ was denoted as $\mathcal{Q}_\epsilon$ there, and their $|S_\epsilon\rangle$ corresponds to our $\frac{1}{2}\mathcal{Q}^\epsilon|\mathcal{I}\rangle$.) It can easily be seen that this $\mathcal{Q}^A_\epsilon$ does annihilate the identity (hence the superscript A), unlike the $\mathcal{Q}^\epsilon$ defined in (5.14). By definition (4.71) of $\mathcal{Q}^A_\epsilon$, we have

$$2(\langle\mathcal{I}|\mathcal{Q}^A_\epsilon)|\phi\rangle = \langle f_\mathcal{I}\circ(\mathcal{Q}^\epsilon - I\circ\mathcal{Q}^\epsilon)\ f_\mathcal{I}\circ\phi(0)\rangle = 0\,, \tag{4.73}$$

because $f_\mathcal{I}\circ I(z) = f_\mathcal{I}(z)$ as stated in (4.17).

5. Possible derivation of VSFT from OSFT

We begin this section by reviewing the original proposal for vacuum string field theory (VSFT) made by Gaiotto, Rastelli, Sen and Zwiebach in refs. [1, 5, 11]. The interacting field theory action describing the dynamics of bosonic open strings on a single D25-brane is given by [18]

$$S_W(\Phi) = -\frac{1}{g_o^2}\left[\frac{1}{2}\langle\Phi,Q_B\Phi\rangle + \frac{1}{3}\langle\Phi,\Phi*\Phi\rangle\right], \tag{5.1}$$

where g_o is the open string coupling constant, Q_B is the BRST charge associated with the background of the D25-brane, $|\Phi\rangle$ is the string field represented by a state of ghost number

1 in the matter-ghost conformal field theory (CFT), and $\langle A, B\rangle$ denotes the BPZ inner product of two states $|A\rangle$ and $|B\rangle$. The precise definition of the $*$-product among string fields has been given in section 2. There is now strong evidence [24, 25] that the space of string field is big enough to contain the non-perturbative tachyon vacuum configuration which is expected to represent the closed string vacuum left after the unstable D-brane has disappeared. We denote by $|\Phi_0\rangle$ the tachyon vacuum configuration which is a solution to the equation of motion

$$Q_B|\Phi_0\rangle + |\Phi_0 * \Phi_0\rangle = 0\,, \tag{5.2}$$

although the exact form of it is still unknown. Expanding the string field about the tachyon vacuum as $\Phi = \Phi_0 + \widetilde{\Phi}$, we can rewrite the action (5.1) in the form

$$\widetilde{S}_V(\widetilde{\Phi}) \equiv S_W(\Phi_0+\widetilde{\Phi}) - S_W(\Phi_0) = -\frac{1}{g_o^2}\left[\frac{1}{2}\langle\widetilde{\Phi}, Q\widetilde{\Phi}\rangle + \frac{1}{3}\langle\widetilde{\Phi}, \widetilde{\Phi} * \widetilde{\Phi}\rangle\right], \tag{5.3}$$

where the new kinetic operator Q has been defined by

$$Q|\widetilde{\Phi}\rangle = Q_B|\widetilde{\Phi}\rangle + |\Phi_0 * \widetilde{\Phi}\rangle + |\widetilde{\Phi} * \Phi_0\rangle\,. \tag{5.4}$$

Though Q is expected to have vanishing cohomology, which means that there are no physical perturbative excitations of open strings around the tachyon vacuum, we are compelled to fall back on a numerical analysis in the level truncation scheme in order to investigate it directly [26, 27]. Instead of doing so, Rastelli, Sen and Zwiebach (henceforth RSZ) proposed in [1] that one should perform a field redefinition of the type

$$\Psi = e^{-K}\widetilde{\Phi} \tag{5.5}$$

which leaves the form of the cubic term unchanged, while it does alter the kinetic operator Q into a simpler one $\mathcal{Q} \equiv e^{-K}Qe^{K}$. RSZ conjectured that we can take $\mathcal{Q}$ to be constructed purely out of ghost fields, and to satisfy some requisite properties. This way, we have been led to the following vacuum string field theory action

$$S_V(\Psi) = -\frac{1}{g_o^2}\left[\frac{1}{2}\langle\Psi, \mathcal{Q}\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi * \Psi\rangle\right]. \tag{5.6}$$

Afterward, the formulation of VSFT was completed by specifying the precise form of $\mathcal{Q}$ [11]. The authors of [11] chose a local insertion of a c-ghost at the string midpoint,

$$\mathcal{Q} = \frac{(g_o^2\kappa_0)^{1/3}}{2i}\left(c(i) - c(-i)\right), \tag{5.7}$$

as a plausible candidate for the kinetic operator characterizing VSFT. It was argued that the normalization constant $(g_o^2\kappa_0)^{1/3}$ should be infinite for classical solutions to have non-vanishing action. Evidence[9] for this special choice (5.7) of $\mathcal{Q}$ was given by explaining how such an operator could arise as a result of singular field redefinition.

[9] It was also shown in [11] that their choice (5.7) seemed to agree numerically with another candidate for $\mathcal{Q}$ which had been found by Hata and Kawano in [8] by requiring that the solution to the equation of motion *in* the Siegel gauge should also solve the full set of equations of motion (i.e. without gauge-fixing).

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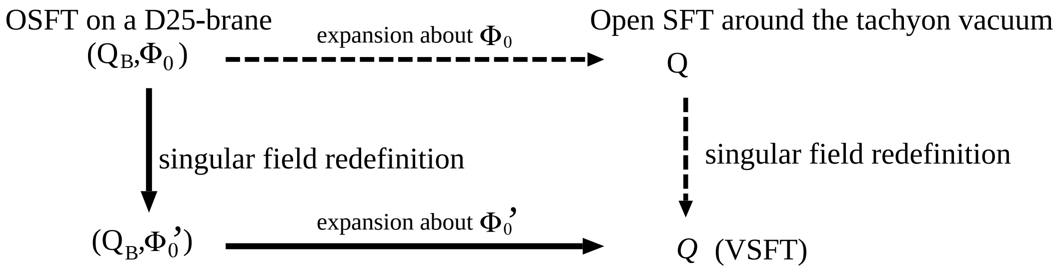

Figure 11: Each corner corresponds to the open string field theory specified by the displayed kinetic operator (Q_B, Q or $\mathcal{Q}$). Φ_0 and Φ_0' denote the tachyon vacuum configurations in the respective theories.

The VSFT scenario is summarized in figure 11. As described in the last paragraph, the proposal of RSZ proceeds along the dashed arrows: perform a field redefinition *after* expanding around the tachyon vacuum solution. Here we assume that we can find a suitable field redefinition such that the order of these two operations is reversed (in other words, that this diagram is commutative). Moreover, suppose that the required field redefinition takes the form

$$\Phi' = e^{-K}\Phi\,, \tag{5.8}$$

where the Grassmann-even operator K of ghost number 0 satisfies

$${}_{12}\langle R|\left(K^{(1)}+K^{(2)}\right) = 0\,, \qquad \text{(BPZ-odd)} \tag{5.9}$$

$${}_{123}\langle V_3|\left(K^{(1)}+K^{(2)}+K^{(3)}\right) = 0\,, \tag{5.10}$$

$$[Q_B, K] = 0\,, \tag{5.11}$$

in the operator language. Properties (5.9) and (5.10) guarantee that K acts as a derivation of the $*$-algebra. Examples of such an operator are provided by

$$K_n = L_n - (-1)^n L_{-n}\,, \tag{5.12}$$

which generate conformal transformations that leave the string midpoint fixed. Indeed, a candidate field redefinition considered in [11] was induced by a reparametrization $f:\sigma\to\sigma'$ of the open string coordinate σ that satisfied $f(\pi-\sigma)=\pi-f(\sigma)$ for $0\le\sigma\le\pi$, which implied $f(\pi/2)=\pi/2$ (midpoint fixed). We do not know whether every operator K satisfying the requirements (5.9)–(5.11) can be written as a linear combination of K_n's. In any case, it will suffice to postulate such an operator K which generates the appropriate field redefinition (5.8). Thanks to the properties (5.9)–(5.11), the form of the original open string field theory action (5.1) is not affected by the field redefinition (5.8) at all.[10] On the other

[10] Since it leaves the action invariant, the field redefinition (5.8) may be expressed as a gauge transformation with a suitable choice of gauge parameter. In fact, it was shown in [27] and by Hata that K_n's given in (5.12) generated gauge transformations when they acted on the tachyon vacuum solution Φ_0. Hence, in case K is expressed as a linear combination of K_n's, it follows that the new tachyon vacuum configuration Φ_0' is actually gauge-equivalent to the original solution Φ_0. For our purpose, however, it does not matter whether the field redefinition (5.8) in question belongs to gauge degrees of freedom or not.

hand, the expression of the tachyon vacuum configuration in the new frame, denoted by Φ_0', in general takes a completely different form than that Φ_0 in the original frame. In particular, recalling that we needed a singular field redefinition to achieve the specific form (5.7) of $\mathcal{Q}$ in [11], it is natural to think that our field redefinition (5.8) should also be singular. Then, starting from the tachyon vacuum solution Φ_0 in the original frame, which is expected to be regular if we remember the results obtained in the level truncation analysis [24], the new configuration $\Phi_0' = e^{-K}\Phi_0$ must be quite singular. Another important property that the solution Φ_0' must have is that the purely ghostly kinetic operator $\mathcal{Q}$ has to arise directly from the expansion of the action around Φ_0'. This is because we no longer have degrees of freedom of field redefinition. To meet these two requirements, we anticipate that the solution will be of the form

$$|\Phi_0'\rangle = -Q_L|\mathcal{I}\rangle + \lim_{\epsilon\to 0}\frac{a}{2}\mathcal{Q}^\epsilon|\mathcal{I}\rangle\,, \tag{5.13}$$

where Q_L defined in (4.30) is the BRST current $j_B(z)$ integrated over left-half of the open string, a is a normalization constant, and $\mathcal{Q}^\epsilon$ is defined by[11]

$$\mathcal{Q}^\epsilon = \frac{1}{2i}\left(e^{-i\epsilon}c(ie^{i\epsilon}) - e^{i\epsilon}c(-ie^{-i\epsilon})\right)\,, \tag{5.14}$$

i.e. a c-ghost inserted *near* the midpoint of the open string, and it smoothly connects to the conjectured kinetic operator $\mathcal{Q}$ (5.7) in the limit $\epsilon\to 0$ when it is acting on a Fock space state. $\mathcal{Q}^\epsilon$ defined this way is intended *not* to annihilate the state $|\mathcal{I}\rangle$ for any value of ϵ so that $\mathcal{Q}^\epsilon|\mathcal{I}\rangle$ becomes singular in the limit $\epsilon\to 0$. We will show that $|\Phi_0'\rangle$ given in (5.13) really solves the equation of motion (5.2), and that this solution also possesses the second property that it gives rise to the desired ghostly kinetic operator $\mathcal{Q}$ in a regularized form when the action (5.1) is expanded around it. In fact, it has long been discussed that the configurations of the form (5.13) actually solve the equation of motion (5.2), especially in the context of purely cubic string field theory [14]. The argument there, however, heavily relied on the existence and the formal properties of the identity element $\mathcal{I}$ of the $*$-algebra. In this paper we have concretely defined the "identity" state as the wedge state of an angle 2π (see section 2 for details), and have not required $\mathcal{I}$ to satisfy $\mathcal{I}*A = A*\mathcal{I} = A$ for every state A. Since the inner product of $|\mathcal{I}\rangle$ with any Fock space state $|\phi\rangle$ has been defined, we consider $|\mathcal{I}\rangle$ not as a formal object but as a real one. In this sense, our present work may be considered as an attempt to put the arguments using the identity string field $\mathcal{I}$ on a firm basis. With this in mind, we want to argue that the solution (5.13) may connect the ordinary cubic open string field theory on a D25-brane background to vacuum string field theory.

From now on, we will omit the symbol $\lim_{\epsilon\to 0}$ and write simply as

$$|\Phi_0'\rangle = -Q_L|\mathcal{I}\rangle + \frac{a}{2}\mathcal{Q}^\epsilon|\mathcal{I}\rangle\,, \tag{5.15}$$

with the understanding that we take such a limit after the entire calculation is over. First of all, we must verify that this configuration really solves the equation of motion

$$Q_B|\Phi_0'\rangle + |\Phi_0' * \Phi_0'\rangle = 0\,. \tag{5.16}$$

[11] Note that our $\mathcal{Q}^\epsilon$ is different from $\mathcal{Q}_\epsilon$ defined in [11, eq. (2.24)].

To see this, we take the BPZ inner product of the left-hand side of (5.16) with an arbitrary Fock space state $|\phi\rangle$. The contribution from the second term of it is

$$\begin{aligned}\langle\phi,\Phi_0'*\Phi_0'\rangle &= \langle\phi,Q_L\mathcal{I}*Q_L\mathcal{I}\rangle+\frac{a^2}{4}\langle\phi,\mathcal{Q}^\epsilon\mathcal{I}*\mathcal{Q}^\epsilon\mathcal{I}\rangle- \\ &\quad -\frac{a}{2}\left(\langle\phi,Q_L\mathcal{I}*\mathcal{Q}^\epsilon\mathcal{I}\rangle+\langle\phi,\mathcal{Q}^\epsilon\mathcal{I}*Q_L\mathcal{I}\rangle\right). \end{aligned} \tag{5.17}$$

The second term of (5.17) vanishes because of eq. (4.23). To the first term we apply the partial integration formulae we have derived in subsection 4.2. On one hand, eq. (4.63) for $\mathcal{O}=Q_L$ allows us to write

$$\langle\phi,Q_L\mathcal{I}*Q_L\mathcal{I}\rangle=\langle\phi,Q_RQ_L\mathcal{I}*\mathcal{I}\rangle\,. \tag{5.18}$$

On the other hand, thanks to the relation[12] $\langle\mathcal{I}|Q_B=\langle\mathcal{I}|(Q_L+Q_R)=0$ and eq. (4.64) for $\mathcal{O}=Q_L$ we get

$$\langle\phi,Q_L\mathcal{I}*Q_L\mathcal{I}\rangle=-\langle\phi,Q_R\mathcal{I}*Q_L\mathcal{I}\rangle=\langle\phi,\mathcal{I}*Q_LQ_L\mathcal{I}\rangle\,. \tag{5.19}$$

Using eq. (4.10) to treat $\mathcal{I}$ as the identity under the $*$-product, (5.18)+(5.19) becomes

$$2\langle\phi,Q_L\mathcal{I}*Q_L\mathcal{I}\rangle=\langle\phi,Q_BQ_L\mathcal{I}\rangle=-\langle\phi,Q_LQ_B\mathcal{I}\rangle=0$$

because Q_B anticommutes with Q_L (4.29) and Q_B kills the identity. Therefore the first term of (5.17) vanishes as well. Utilizing the partial integration formulae again, the second line of eq. (5.17) can be taken to the form

$$\begin{aligned}-\langle\phi,Q_R\mathcal{I}*\mathcal{Q}^\epsilon\mathcal{I}\rangle+\langle\phi,\mathcal{Q}^\epsilon\mathcal{I}*Q_L\mathcal{I}\rangle &= \langle\phi,\mathcal{I}*Q_L\mathcal{Q}^\epsilon\mathcal{I}\rangle+\langle\phi,Q_R\mathcal{Q}^\epsilon\mathcal{I}*\mathcal{I}\rangle \\ &= \langle\phi,Q_B\mathcal{Q}^\epsilon\mathcal{I}\rangle\,,\end{aligned}$$

where eqs. (4.63), (4.64) and (4.10) have been used. Thus we have found

$$\langle\phi,\Phi_0'*\Phi_0'\rangle=-\frac{a}{2}\langle\phi,Q_B\mathcal{Q}^\epsilon\mathcal{I}\rangle\,. \tag{5.20}$$

The first term of (5.16) gives

$$\begin{aligned}\langle\phi,Q_B\Phi_0'\rangle &= \left\langle\phi,Q_B\left(-Q_L\mathcal{I}+\frac{a}{2}\mathcal{Q}^\epsilon\mathcal{I}\right)\right\rangle=\langle\phi,Q_LQ_B\mathcal{I}\rangle+\frac{a}{2}\langle\phi,Q_B\mathcal{Q}^\epsilon\mathcal{I}\rangle \\ &= \frac{a}{2}\langle\phi,Q_B\mathcal{Q}^\epsilon\mathcal{I}\rangle\,.\end{aligned} \tag{5.21}$$

Adding (5.20) and (5.21), we have at last reached

$$\langle\phi,Q_B\Phi_0'+\Phi_0'*\Phi_0'\rangle=0\,, \tag{5.22}$$

which means that the equation of motion (5.16) is indeed solved by the classical configuration (5.15), at least in a weak sense. Namely, though we have shown that the left-hand

[12] Although we do not give explicit proofs here, we can show not only $\langle\mathcal{I}|Q_B|\phi\rangle=0$ (4.31) but also $\langle\phi,Q_B\mathcal{I}*Q_L\mathcal{I}\rangle=0$, $\langle\phi,Q_B\mathcal{I}*\mathcal{Q}^\epsilon\mathcal{I}\rangle=0$, and $\langle\phi,Q_B\mathcal{I}*\psi\rangle=0$, which will be sufficient for later purposes, by considering the integration contour of Q_B just as in subsection 4.2.

side of eq. (5.16) has vanishing inner product with *any Fock space state* in a narrow sense, we have not examined whether it does vanish for *all* states including the identity state and so on. (cf. [13]) Even if the answer turns out to be no, we will take the standpoint that we may consider the state which solves the equation of motion *only* in a weak sense to be a respectable solution.

We want to call the reader's attention to the fact that if the coefficient of $Q_L\mathcal{I}$ in (5.15) were not -1, the cancellation between (5.20) and (5.21) would be incomplete, accordingly the equation of motion (5.16) would not be satisfied. Thus the equation of motion requires this coefficient to be -1, as long as $a \neq 0$. To the contrary, the value of a (the coefficient of $\mathcal{Q}^\epsilon\mathcal{I}$) remains completely undetermined by the equation of motion. This issue will be discussed in section 6.

Now that we have shown $|\Phi_0'\rangle$ given in (5.15) is a classical solution, let us expand the string field Φ' around this solution as $\Phi' = \Phi_0' + \Psi$ and substitute it into the action (5.1) with Φ replaced by Φ'. This gives

$$S_W(\Phi_0' + \Psi) = -\frac{1}{g_o^2}\sum_{n=0}^{3} S_n\,,$$

where

$$\begin{aligned}
S_0 &= -g_o^2 S_W(\Phi_0') \\
&= \frac{a^2}{8}\langle \mathcal{Q}^\epsilon\mathcal{I}, Q_B\mathcal{Q}^\epsilon\mathcal{I}\rangle + \frac{1}{2}\langle Q_L\mathcal{I}, Q_B Q_L\mathcal{I}\rangle - \frac{a}{2}\langle \mathcal{Q}^\epsilon\mathcal{I}, Q_B Q_L\mathcal{I}\rangle - \\
&\quad - \frac{1}{3}\langle Q_L\mathcal{I}, Q_L\mathcal{I} * Q_L\mathcal{I}\rangle + \frac{a}{2}\langle \mathcal{Q}^\epsilon\mathcal{I}, Q_L\mathcal{I} * Q_L\mathcal{I}\rangle - \\
&\quad - \frac{a^2}{4}\langle Q_L\mathcal{I}, \mathcal{Q}^\epsilon\mathcal{I} * \mathcal{Q}^\epsilon\mathcal{I}\rangle + \frac{a^3}{24}\langle \mathcal{Q}^\epsilon\mathcal{I}, \mathcal{Q}^\epsilon\mathcal{I} * \mathcal{Q}^\epsilon\mathcal{I}\rangle\,, && (5.23)\\
S_1 &= \langle \Psi, Q_B\Phi_0' + \Phi_0' * \Phi_0'\rangle\,, && (5.24)\\
S_2 &= \frac{1}{2}\langle \Psi, Q_B\Psi\rangle - \langle Q_L\mathcal{I}, \Psi * \Psi\rangle + \frac{a}{2}\langle \mathcal{Q}^\epsilon\mathcal{I}, \Psi * \Psi\rangle\,, && (5.25)\\
S_3 &= \frac{1}{3}\langle \Psi, \Psi * \Psi\rangle\,. && (5.26)
\end{aligned}$$

If we restrict the fluctuation Ψ to lying in the Fock space,[13] then the term S_1 which is linear in Ψ vanishes because of the equation of motion (5.22). The term S_2 quadratic in Ψ can be arranged as

$$\begin{aligned}
S_2 = &\ \frac{1}{2}\left(\langle \Psi, Q_B\Psi\rangle - \langle \Psi, Q_L\mathcal{I} * \Psi\rangle - \langle \Psi, \Psi * Q_L\mathcal{I}\rangle\right) + \\
&+ \frac{a}{4}\left(\langle \Psi, \Psi * \mathcal{Q}^\epsilon\mathcal{I}\rangle + \langle \Psi, \mathcal{Q}^\epsilon\mathcal{I} * \Psi\rangle\right). \qquad (5.27)
\end{aligned}$$

Using the relation $\langle \Psi, Q_L\mathcal{I} * \Psi\rangle = -\langle \Psi, Q_R\mathcal{I} * \Psi\rangle$ and the partial integration formulae (4.65) and (4.66), twice the first line (5.27) becomes

$$\langle \Psi, Q_B\Psi\rangle - \langle \Psi, \mathcal{I} * Q_L\Psi\rangle - \langle \Psi, Q_R\Psi * \mathcal{I}\rangle = \langle \Psi, Q_B\Psi\rangle - \langle \Psi, (Q_L + Q_R)\Psi\rangle = 0\,,$$

[13] Due to this restriction, the situation might become subtle when we consider non-perturbative classical solutions in the theory expanded around Φ_0'.

where we have used the relation (4.1) which allows us to treat $\mathcal{I}$ as the identity element under the $*$-product. On the other hand, the second line (5.27) is nothing other than the expression (4.72). Thus we have found

$$S_2 = \frac{a}{2}\langle \Psi, \mathcal{Q}_\epsilon^A \Psi \rangle \,. \tag{5.28}$$

Collecting the above results, the new action describing the theory around Φ_0' becomes

$$\overline{S}_V(\Psi) \equiv S_W(\Phi_0' + \Psi) - S_W(\Phi_0') = -\frac{1}{g_o^2}\left[\frac{a}{2}\langle \Psi, \mathcal{Q}_\epsilon^A \Psi\rangle + \frac{1}{3}\langle \Psi, \Psi * \Psi\rangle\right]. \tag{5.29}$$

Given that the fluctuation field Ψ lives in the Fock space, $\epsilon \to 0$ limit can be taken smoothly. Defining $\lim_{\epsilon\to 0} a\mathcal{Q}_\epsilon^A \equiv \mathcal{Q}$ with an unconventional normalization factor, the above action does seem to agree with the conjectured VSFT action (5.6). Moreover, the kinetic operator $\mathcal{Q}$ has arisen with being regularized in such a way that it should annihilate the identity state $\mathcal{Q}|\mathcal{I}\rangle = \lim_{\epsilon\to 0} a\mathcal{Q}_\epsilon^A|\mathcal{I}\rangle = 0$, as shown in (4.73).

To sum up, what we have shown is that the classical configuration (5.13) gives a solution to the equation of motion (5.16) derived from the action (5.1) describing the original D25-brane, and that the string field theory action expanded around this solution coincides with the vacuum string field theory action proposed in [11], *up to* the normalization of the kinetic operator.

6. Discussion

In this paper, we have focused our attention mainly on the conformal field theory description of the identity string field $\mathcal{I}$. By giving the identity state a precise definition as the wedge state of an angle 2π, we have avoided falling into formal arguments which have been given so far. Furthermore, the generalized gluing and resmoothing theorem has provided us with a useful computational tool which is suitable for dealing with the wedge-type surface states. Based on the results obtained from the CFT calculations, we have specified a solution around which the theory is described by the vacuum string field theory action proposed in [11]. We guess that this solution represents the tachyon vacuum in some singular coordinate system. In the rest of this concluding section we discuss some open questions and give some remarks on further studies.

6.1 Uniqueness of the solution?

In the previous sections, we have shown that the classical configuration

$$\Phi_0'(t,a) = -tQ_L\mathcal{I} + \frac{a}{2}\mathcal{Q}^\epsilon\mathcal{I} \tag{6.1}$$

solves the equation of motion (5.16) if $t = 1$. As remarked in section 5, however, the equation of motion does not constrain the value of a at all. This seems problematic because we

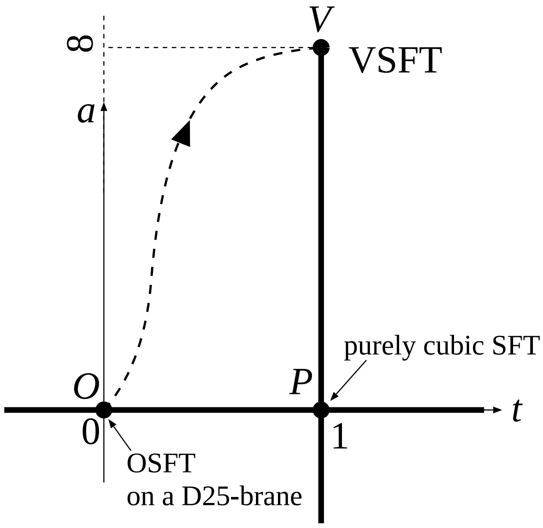

Figure 12: Two-dimensional configuration space of Φ'_0 (6.1). At every point on the bold lines $\Phi'_0(t,a)$ gives a solution to the equation of motion.

then have a one-parameter family of solutions, which prevents us from interpreting $\Phi'_0(1,a)$ as the tachyon vacuum solution. Although we have no appropriate way of determining the value of a uniquely, it would be worthwhile to notice that in order for the action (5.29) around Φ'_0 to agree with the VSFT action (5.6) even including their overall normalizations, we must have $a=(g_o^2\kappa_0)^{1/3}$. Since there is an argument suggesting that one needs to take κ_0 to be infinite, we shall tentatively assume that only $\Phi'_0(t=1,a=\infty)$ corresponds to the tachyon vacuum solution.

In addition, if we set $a=0$, $\Phi'_0(t,0)=-tQ_L\mathcal{I}$ gives a solution to the equation of motion (5.16) for any value of t. This is because each term of the left-hand side of (5.16) vanishes independently of each other according to the argument given in section 5. Note that $(t,a)=(1,0)$ leads to the purely cubic string field theory [14]. The configuration space of Φ'_0 parametrized by (t,a) is drawn in figure 12. We see that the set of all solutions of the form (6.1) consists of two branches which meet at the purely cubic string field theory point.

Let us consider the tachyon condensation process which is now supposed to roll down from $(t,a)=(0,0)$ to $(1,\infty)$. Since it sounds unlikely that the string field Φ' keeps on satisfying the equation of motion during the condensation process, we assume that it proceeds along the dashed curve OV shown in figure 12, rather than along the bold lines OPV. In the vicinity of the VSFT point, we introduce a new parameter Λ through the relation

$$t=1-\frac{a}{\Lambda}\,. \tag{6.2}$$

Of course, $t=1$ corresponds to $\Lambda\to\infty$. On the dashed curve OV, t and a are implicitly determined as functions of Λ, so we write them as $t=t(\Lambda), a=a(\Lambda)$. Hence we can think of Λ as the coordinate on the curve OV. Now consider the configuration $\Phi'_0(1-\frac{a(\Lambda)}{\Lambda},a(\Lambda))$ and expand the original action (5.1) around it, even though it *does not* correspond to a classical solution. The resulting action is found to be

$$\widehat{S}_V(\Psi;\Lambda) \equiv S_W(\Phi_0'+\Psi) - S_W(\Phi_0') = -\frac{1}{g_o^2}\left[\frac{a(\Lambda)^2}{2\Lambda}\langle\Psi, Q_B\mathcal{Q}^\epsilon\mathcal{I}\rangle + \frac{a(\Lambda)}{2\Lambda}\langle\Psi, Q_B\Psi\rangle + \right.$$
$$\left. + \frac{a(\Lambda)}{2}\langle\Psi, \mathcal{Q}_\epsilon^A\Psi\rangle + \frac{1}{3}\langle\Psi, \Psi*\Psi\rangle\right]. \quad (6.3)$$

Below, we simply ignore the first term which is linear in Ψ and would not exist if Φ_0' were a solution. Upon fixing in the Siegel gauge, the remaining terms are written as

$$\widehat{S}_V(\Psi;\Lambda) = -\frac{1}{g_o^2}\left[\frac{a(\Lambda)}{2}\left\langle\Psi, c_0\left(1+\frac{L_0^{\text{tot}}}{\Lambda}\right)\Psi\right\rangle + \frac{1}{3}\langle\Psi,\Psi*\Psi\rangle\right]. \quad (6.4)$$

This form closely resembles the "regularized VSFT action" considered in [11] to regularize the singular nature of vacuum string field theory. As the authors of [11] mentioned, by taking the regularization parameter Λ to be infinite (hence $t=1$), $\widehat{S}_V$ reduces to the singular VSFT action (5.29) in a gauge-fixed form. There is, however, a wide difference between ours (6.4) and theirs. Their regularization is achieved by replacing the singular reparametrization with a nearly singular one leading to an equivalent theory, whereas our regularization represented by Λ corresponds to the deformation of the classical configuration around which we are going to expand the action, which of course is expected to give rise to an inequivalent theory. In particular, we have to take into account the term linear in Ψ since for finite Λ, Φ_0' is not even a solution. Anyway, it is obvious that we need a better understanding of the regularization procedure to find a precise relation between ours and theirs.

6.2 D25-brane tension

In spite of the fact that we started from the well-established cubic open string field theory on a D25-brane, we are unable to calculate the energy density associated with the solution. This problem is closely related to the issue mentioned in the last subsection, namely that the parameter a is left unfixed. Thinking in reverse order, it might be possible to determine a by requiring that the value of the action $S_W(\Phi_0'(t=1,a))$ be equal to the negative of the energy of a single D25-brane, $-V_{26}\mathcal{T}_{25}$, where V_{26} represents the volume of the 26-dimensional spacetime and $\mathcal{T}_{25}$ denotes the tension of the D25-brane. Using the relation [28]

$$\mathcal{T}_{25} = \frac{1}{2\pi^2 g_o^2}, \quad (6.5)$$

the equation we want to solve becomes

$$-g_o^2 S_W(\Phi_0'(1,a)) \equiv S_0(a) = \frac{V_{26}}{2\pi^2}, \quad (6.6)$$

where the expression for $S_0(a)$ has been given in eq. (5.23). If we could solve this equation with respect to a, the result of which is expected to diverge in some sense, then there would remain no unfixed parameter in the theory around Φ_0' (and presumably in VSFT).

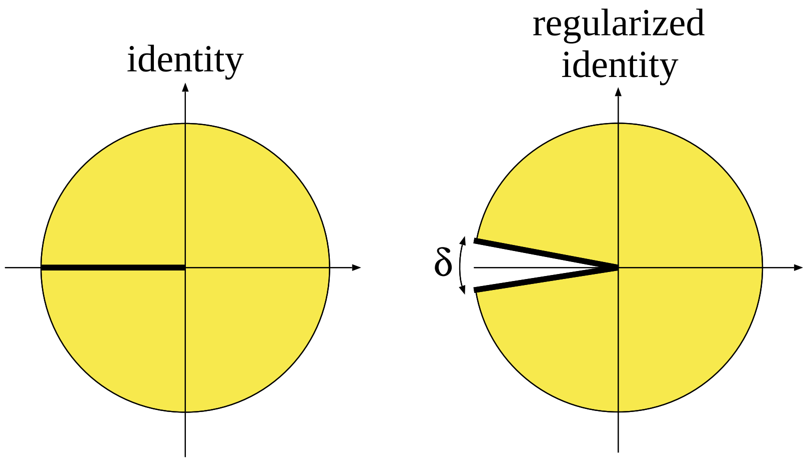

Figure 13: Deformation of the identity state.

But unfortunately, it has turned out that $S_0(a)$ is quite badly-behaved. For example, we have obtained[14]

$$\langle \mathcal{Q}^\epsilon \widetilde{\mathcal{I}}_\delta, Q_B \mathcal{Q}^\epsilon \widetilde{\mathcal{I}}_\delta \rangle = -\delta^2 \sin^2 \epsilon \left[\frac{1}{2} \left\{ \left(\tan \frac{\epsilon}{2} \right)^{2/\delta} + \left(\tan \frac{\epsilon}{2} \right)^{-2/\delta} \right\} + 3 \right] V_{26} \,, \tag{6.7}$$

where δ is a regularization parameter introduced to deform the identity state in a way indicated in figure 13. To be more specific, the regularized identity state is defined by

$$\left\langle \widetilde{\mathcal{I}}_\delta, \mathcal{O} \right\rangle = \left\langle \widetilde{f}_{\mathcal{I}_\delta} \circ \mathcal{O}(0) \right\rangle , \qquad \text{with } \widetilde{f}_{\mathcal{I}_\delta}(z) = h^{-1} \left(h(z)^{\frac{2}{1+\delta}} \right) . \tag{6.8}$$

This regularization is necessary because if we naïvely apply the gluing theorem to the expression of the form $\langle \mathcal{I}, \mathcal{I} \rangle$ then we are left with an ill-defined expression. This can be seen from the fact that we cannot take the $\delta \to 0$ limit in eq. (6.7). The evaluation of $S_0(a)$ has not been successful up to now, and is under investigation [29].

6.3 Other solutions?

Since the action $\overline{S}_V(\Psi)$ in (5.29) has been obtained by expanding the original action $S_W(\Phi')$ (5.1) around the classical solution $\Phi'_0 = -Q_L \mathcal{I} + \frac{a}{2} \mathcal{Q}^\epsilon \mathcal{I}$, we can recover the action $S_W(\Phi')$ by re-expanding the "VSFT action" $\overline{S}_V(\Psi)$ around the solution

$$\Psi_0 = -\Phi'_0 = Q_L \mathcal{I} - \frac{a}{2} \mathcal{Q}^\epsilon \mathcal{I} \tag{6.9}$$

as $\Psi = \Psi_0 + \Phi'$, and it is obvious that this solution represents a single D25-brane. As remarked in a footnote of section 5, however, the action $\overline{S}_V(\Psi)$ given in (5.29) has been shown to be valid only for Ψ lying in the Fock space. Since the above solution Ψ_0 is outside the Fock space in a narrow sense, it may not be correct to suppose the theory around Φ'_0 to be literally described by the action $\overline{S}_V(\Psi)$ in this case. Nevertheless, we shall have arguments below on the assumption that the theory around Φ'_0 is really governed by $\overline{S}_V(\Psi)$.

[14] Note that this quantity would vanish if we naïvely use equation of motion for $\delta = 0$.

If we are to regard $\overline{S}_V(\Psi)$ as representing the vacuum string field theory under an appropriate choice of the normalization constant a, then we must be able to construct other D-brane solutions than the original single D25-brane, that is to say, lower-dimensional D-branes, multiple D-branes, and so forth. However, as pointed out in [2], it seems difficult to produce spatially localized solutions if we adhere to the identity-type solution of the form (6.9). It is true that once a D25-brane (6.9) has been constructed lower-dimensional D-branes could be obtained as lump solutions on the D25-brane via tachyon condensation at least in the level truncation scheme [30], but it is obviously desirable to build them directly.

In this connection, we also give a brief comment on the possible relation to the projector-type[15] solutions which have been examined in the context of vacuum string field theory [2, 11]. If both the identity-type solution (6.9) and one of the spatially-independent projector-type solutions represent one and only D25-brane, they should be related to each other through some gauge transformation. We do not know whether such a surprising thing could happen.

6.4 Mystery on $c_0\mathcal{I}$ revisited

The following is a well-known argument: Since c_0 acts as a derivation of the $*$-algebra, it follows that

$$c_0 A = c_0(\mathcal{I} * A) = (c_0\mathcal{I}) * A + \mathcal{I} * (c_0 A) = (c_0\mathcal{I}) * A + c_0 A \tag{6.10}$$

for a *true identity state* $\mathcal{I}$ and for any A. For this equation to hold, we must have $c_0\mathcal{I} = 0$, which is not the case [19]. Hence one is tempted to conclude that there does not exist such a true identity state. Though this conclusion may be true in a sense, the above argument is too strong. In fact, it is sufficient to have $(c_0\mathcal{I}) * A = 0$ rather than $c_0\mathcal{I} = 0$. That is to say, there is no problem in eq. (6.10) if we show that $c_0\mathcal{I}$ gives zero whenever it is $*$-multiplied by an arbitrary element. In the operator language, this statement is equivalent to the relation ${}_1\langle\mathcal{I}|c_0^{(1)}|V_3\rangle_{123} = 0$, and it was shown in [9] that it is indeed true.

From the point of view of our present paper, we can calculate

$$\langle\phi, (c_0\mathcal{I}) * A\rangle + \langle\phi, \mathcal{I} * (c_0 A)\rangle \tag{6.11}$$

and see if it is equal to $\langle\phi, c_0 A\rangle$. For simplicity we take $|\phi\rangle$ and $|A\rangle$ to be Fock space states. The second term of (6.11) reduces to $\langle\phi, c_0 A\rangle$ thanks to the relation (4.1). We have calculated the first term with the help of the gluing theorem and obtained

$$\langle\phi, (c_0\mathcal{I}) * A\rangle = \langle\phi, c_0 A\rangle \neq 0\,, \tag{6.12}$$

contrary to our expectation. In addition, we can easily show $\langle\phi, c_0(\mathcal{I} * A)\rangle = \langle\phi, c_0 A\rangle$ directly. After all, we have found

$$\langle\phi, (c_0\mathcal{I}) * A\rangle + \langle\phi, \mathcal{I} * (c_0 A)\rangle = 2\langle\phi, c_0 A\rangle \neq \langle\phi, c_0 A\rangle = \langle\phi, c_0(\mathcal{I} * A)\rangle\,. \tag{6.13}$$

[15] By the word "projector-type solutions" we mean the (twisted) sliver state and the (twisted) butterfly state as well as the family they belong to.

Since there is no ambiguity in our computational scheme, something must be wrong with eq. (6.10). A possible resolution of this problem would be to think that it depends on the states to be acted on whether the operator c_0 may be regarded as a derivation. We have not reached a definite conclusion concerning this point. To have a better understanding of the identity state, it would be important to examine the action of such outer derivations on $\mathcal{I}$ in more detail.

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